Magneto optic Effects

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Ref: I.D. Vagner *et. al.* Electrodynamics of Magnetoactive Media

EM waves in anisotropic media

Anisotropic properties given by ϵ_{ik} and μ_{ik}

$$D_i = \epsilon_{ik}(\omega)E_k \qquad B_i = \mu_{ik}(\omega)H_k$$

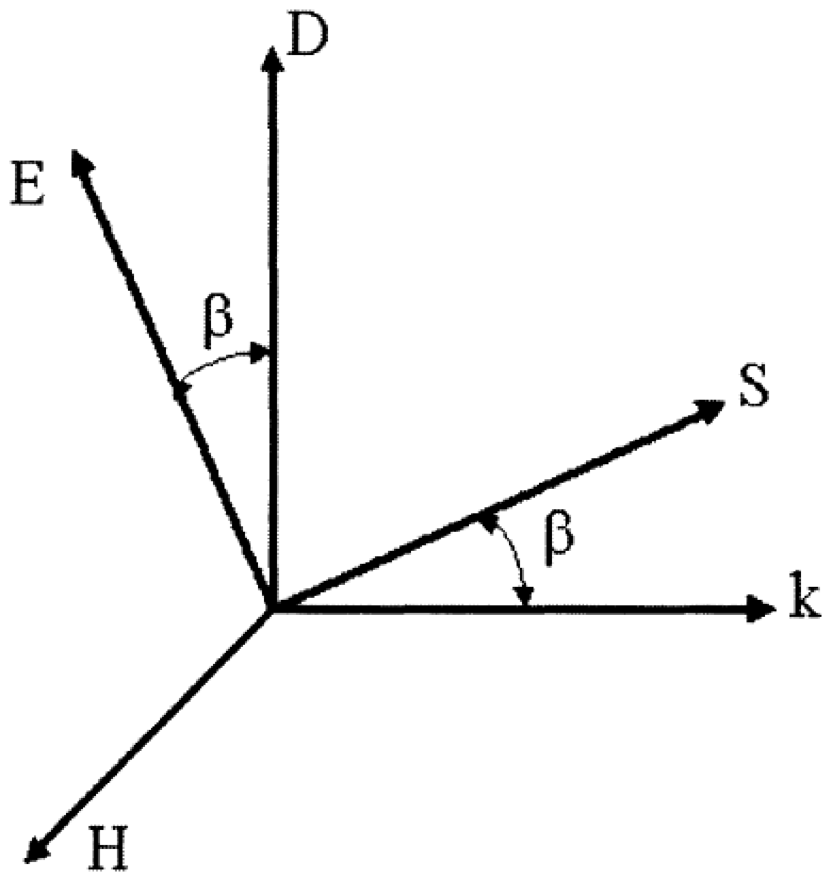
Dimensionless: $$\epsilon^r_{ik} = \frac{\epsilon_{ik}}{\epsilon_0} \qquad \mu^r_{ik} = \frac{\mu_{ik}}{\mu_0}$$

Initially considered nonmagnetic media with no loss

↓ $\mu_{ik} = \mu_0$

↓ $\text{Im}\ \epsilon_{ik} = 0$

Maxwell's eqns,



$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{k} \times \vec{E} = \omega \mu_0 \vec{H}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \Rightarrow -\vec{k} \times \vec{H} = \omega \vec{D}$$

$\Rightarrow \vec{k}, \vec{D}, \vec{H} \rightarrow$ mutually orthogonal

$\vec{H} \perp \vec{E} \Rightarrow \vec{D}, \vec{E}, \vec{k}$ all $\perp \vec{H}$

$\Rightarrow$ must be coplanar

$\vec{E}, \vec{D}, \vec{H}, \vec{k}, \vec{S}$ for a plane wave

In contrast to isotropic media,

$\vec{D}$ and $\vec{H}$ are $\perp \vec{k}$ But $\vec{E}$ is not perpendicular to k

$\vec{S} = \vec{E} \times \vec{H}$ is not $\perp \vec{k}$ $\vec{S}$ is not $\parallel$ to $\vec{k}$

Clearly, $\vec{S}$ is coplanar with $\vec{E}$, $\vec{D}$, $\vec{k}$

Angle between $\vec{S}$ and $\vec{k}$ same as angle between $\vec{D}$ and $\vec{E}$

Define dimensionless vector $\vec{n} \Rightarrow \vec{k} = \frac{\omega}{c}\vec{n}$

Magnitude of $\vec{n}$ in anisotropic medium depends on direction,

while in isotropic medium, $n = \sqrt{\epsilon_r}$ depends only on frequency.

$$\vec{k} \times \vec{E} = \omega\mu_0 \vec{H} \Rightarrow \frac{\omega}{c}\vec{n} \times \vec{E} = \omega\mu_0 \vec{H}$$

$$\Rightarrow \vec{n} \times \vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}}\vec{H}$$

$$-\vec{k} \times \vec{H} = \omega\vec{D} \Rightarrow -\vec{n} \times \vec{H} = \frac{1}{\sqrt{\epsilon_0\mu_0}}\vec{D}$$

$$\begin{aligned} \vec{S} = \vec{E} \times \vec{H} &= \vec{E} \times \sqrt{\frac{\epsilon_0}{\mu_0}}\,[\vec{n} \times \vec{E}] \\ &= \sqrt{\frac{\epsilon_0}{\mu_0}}\left[\vec{n}E^2 - \vec{E}(\vec{n}\,.\,\vec{E})\right] \end{aligned}$$

Unlike the isotropic case now $\vec{n}\,.\,\vec{E}$ does not vanish since in anisotropic medium $\vec{n}$ no longer is perpendicular to $\vec{E}$.

tifr

$$-\vec{n} \times \vec{H} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \vec{D}, \qquad \vec{n} \times \vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{H}$$

$$\Rightarrow -\vec{n} \times \sqrt{\frac{\epsilon_0}{\mu_0}} (\vec{n} \times \vec{E}) = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \vec{D}$$

$$-\left[\vec{n}(\vec{n}\,.\,\vec{E}) - \vec{E} n^2\right] = \frac{\vec{D}}{\epsilon_0}$$

i th eqn, $\quad n^2 E_i - n_i n_k E_k = \epsilon^r_{ik} E_k \quad$ sum over repeated index

For E_i to be nontrivial, we have to demand

$$\left|n^2 \delta_{ik} - n_i n_k - \epsilon^r_{ik}\right| = 0$$

Let x, y, z be the principal axes of the tensor ϵ^r_{ik} with diagonal elements ϵ^r_{xx}, ϵ^r_{yy}, ϵ^r_{zz}

$$\text{Then, } \left|n^2\delta_{ik} - n_i n_k - \epsilon^r_{ik}\right| = 0$$

↓

$$\begin{vmatrix} n^2 - n_x^2 - \epsilon^r_{xx} & -n_x n_y & -n_x n_z \\ -n_x n_y & n^2 - n_y^2 - \epsilon_{yy} & -n_y n_z \\ -n_z n_x & -n_z n_y & n^2 - n_z^2 - \epsilon_{zz} \end{vmatrix} = 0$$

$$n^2 E_i - n_i n_k E_k = \epsilon^r_{ik} E_k$$

$$n^2 E_i - n_i n_k E_k = \epsilon^r_{ik} E_k$$

↓

$$n^2 E_x - n_x(n_x E_x + n_y E_y + n_z E_z) = \epsilon^r_{xx} E_x + \epsilon^r_{xy} E_y + \epsilon^r_{xz} E_z$$

$$\Rightarrow (n^2 - n_x^2 - \epsilon^r_{xx}) E_x - n_x n_y E_y - n_x n_z E_z = 0$$

Henceforth drop subscipt 'r' $n^2 = n_x^2 + n_z^2 + n_z^2$

Determinant,

$$n^2(\epsilon_{xx}n_x^2+\epsilon_{yy}n_y^2+\epsilon_{zz}n_z^2)-\epsilon_{xx}n_x^2(\epsilon_{yy}+\epsilon_{zz})-\epsilon_{yy}n_y^2(\epsilon_{xx}+\epsilon_{zz})-\epsilon_{zz}n_z^2(\epsilon_{xx}+\epsilon_{yy})+\epsilon_{xx}\epsilon_{yy}\epsilon_{zz}=0$$

Referred to as Fresnel eqns

One of the fundamental eqns of crystal optics.

→ Gives the magnitude of the wave vector as a function of direction

$\Rightarrow$ For a given direction$\rightarrow$ a quadratic eqn of n^2 with real coefficients

$\Rightarrow$ two different magnitudes for each direction

Wave vector surface

direction of light rays- group velocity $\dfrac{\partial \omega}{\partial \vec{k}}$

Isotropic media $\vec{k}$ and $\frac{\partial \omega}{\partial \vec{k}}$ same

anisotropic medium$\rightarrow$ not so

direction $\dfrac{\partial \omega}{\partial \vec{k}} \rightarrow \vec{s}$

$\vec{s}-$ ray vector

magnitude $\vec{n}\,.\,\vec{s} = 1$

Direct calculation $\vec{s}\,.\,\vec{H} = 0$ $\quad$ $\vec{s}\,.\,\vec{E} = 0$

Since $\vec{s} \perp \vec{E},\, \vec{H}$

$$\vec{H} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \vec{s} \times \vec{D} \qquad -\vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{s} \times \vec{H}$$

Replacing $\sqrt{\epsilon_0}\vec{E} \leftrightarrow \dfrac{\vec{D}}{\sqrt{\epsilon_0}}, \quad \vec{n} \leftrightarrow \vec{s}, \quad \epsilon^r_{ik} \leftrightarrow \left(\epsilon^r_{ik}\right)^{-1}$

$$\vec{n} \times \vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{H}, \qquad -\vec{n} \times \vec{H} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \vec{D}$$

(Same as ■)

$$D_i = \epsilon_{ik}(\omega) E_k \qquad B_i = \mu_{ik}(\omega) H_k$$

Remains valid under replacement

Apply to Fresnel eqn.,

$$n^2(\epsilon_{xx}n_x^2 + \epsilon_{yy}n_y^2 + \epsilon_{zz}n_z^2) - \epsilon_{xx}n_x^2(\epsilon_{yy} + \epsilon_{zz})$$
$$-\epsilon_{yy}n_y^2(\epsilon_{xx} + \epsilon_{zz}) - \epsilon_{zz}n_z^2(\epsilon_{xx} + \epsilon_{yy}) + \epsilon_{xx}\epsilon_{yy}\epsilon_{zz} = 0$$

$$s^2\left(\frac{1}{\epsilon_{xx}}s_x^2 + \frac{1}{\epsilon_{yy}}s_y^2 + \frac{1}{\epsilon_{zz}}s_z^2\right) - \frac{1}{\epsilon_{xx}}s_x^2\left(\frac{1}{\epsilon_{yy}} + \frac{1}{\epsilon_{zz}}\right)$$
$$-\frac{1}{\epsilon_{yy}}s_y^2\left(\frac{1}{\epsilon_{xx}} + \frac{1}{\epsilon_{zz}}\right) - \frac{1}{\epsilon_{zz}}s_z^2\left(\frac{1}{\epsilon_{xx}} + \frac{1}{\epsilon_{yy}}\right) + \frac{1}{\epsilon_{xx}\epsilon_{yy}\epsilon_{zz}} = 0$$

$$\Rightarrow \quad s^2(\epsilon_{yy}\epsilon_{zz}s_x^2 + \epsilon_{xx}\epsilon_{zz}s_y^2 + \epsilon_{xx}\epsilon_{yy}s_z^2) - s_x^2(\epsilon_{yy} + \epsilon_{zz})$$

Ray surface $\longrightarrow$

$$-s_y^2(\epsilon_{zz} + \epsilon_{xx}) - s_z^2(\epsilon_{xx} + \epsilon_{yy}) + 1 = 0$$

Uniaxial crystal

$$\text{cubic } \epsilon_{ik} = \epsilon\,\delta_{ik}$$

$$\text{uniaxial } \epsilon_{xx} = \epsilon_{yy} = \epsilon_\perp$$

$$\epsilon_{zz} = \epsilon_\parallel$$

Put in Fresnal eqn

$$\begin{aligned}
n^2(\epsilon_\perp(n_x^2+n_y^2)+\epsilon_\parallel n_z^2) - \epsilon_\perp n_x^2(\epsilon_\parallel+\epsilon_\perp) - \epsilon_\perp n_y^2(\epsilon_\parallel+\epsilon_\perp) - \epsilon_\parallel n_z^2(2\epsilon_\perp) + \epsilon_\perp^2\epsilon_\parallel &= 0 \\
n^2(\epsilon_\perp(n_x^2+n_y^2)+\epsilon_\parallel n_z^2) \; - \; \epsilon_\perp\epsilon_\parallel(n_x^2+n_y^2+n_z^2) - (\epsilon_\perp n_x^2+\epsilon_\perp n_y^2+\epsilon_\parallel n_z^2 - \epsilon_\perp\epsilon_\parallel)\,\epsilon_\perp &= 0 \\
\Rightarrow (\epsilon_\perp(n_x^2+n_y^2)+\epsilon_\parallel n_z^2 - \epsilon_\perp\epsilon_\parallel)\,n^2 - (\epsilon_\perp(n_x^2+n_y^2)+\epsilon_\parallel n_z^2 - \epsilon_\perp\epsilon_\parallel)\,\epsilon_\perp &= 0
\end{aligned}$$

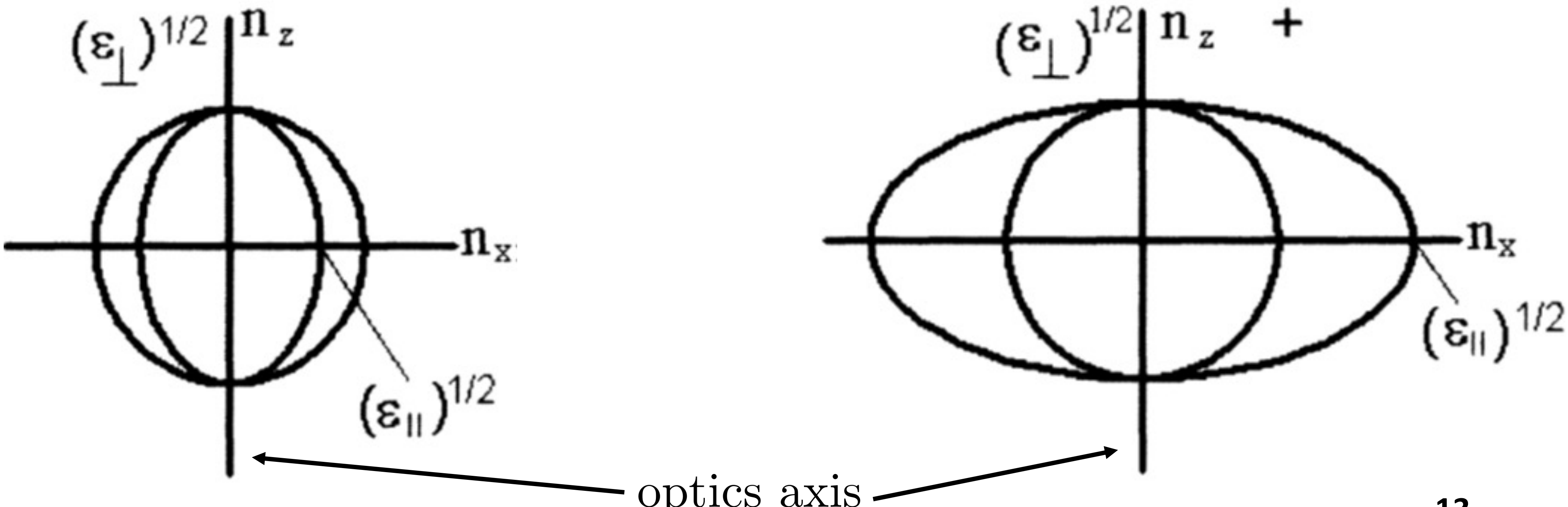
$$\Rightarrow (n^2-\epsilon_\perp)(\epsilon_\perp(n_x^2+n_y^2)+\epsilon_\parallel n_z^2 - \epsilon_\perp\epsilon_\parallel) = 0$$

Quadratic eqn gives two roots

$$\text{(i)}\ n^2 = \epsilon_\perp \ \text{(sphere)}$$

$$\text{(ii)}\ \frac{n_z^2}{\epsilon_\perp} + \frac{n_x^2 + n_y^2}{\epsilon_\parallel} = 1 \ \text{(ellipsoid of rotation)}$$

Two cases: $\epsilon_\perp > \epsilon_\parallel$ −ve crystal $\epsilon_\perp < \epsilon_\parallel$ +ve crystal



Magnitude of the wave vector

$$\text{(a)}\ n^2 = \epsilon_\perp \quad \text{ordinary wave}$$

$$\text{(b)}\ \frac{1}{n^2} = \frac{\sin^2\theta}{\epsilon_\parallel} + \frac{\cos^2\theta}{\epsilon_\perp} \quad \text{Extraordinary waves}$$

$\theta -$ angle between optic axis and $\vec{k}$

Direction of wave vector $\vec{k} = \dfrac{\omega}{c}\vec{n}$

Direction of ray vector not the same as direction of wave vector

But ray vector coplanar with wave vector and optic axis

Let $\theta' \rightarrow$ angle between s and optic axis

$$\tan\theta' = \frac{\epsilon_\perp}{\epsilon_\parallel}\tan\theta$$

Same only when no anisotropic $\dfrac{\epsilon_\perp}{\epsilon_\parallel} = 1$

Magneto optical effect

In presence of a constant magnetic field $\vec{H}$ the tensor ϵ^r_{ik} (we drop r) is no longer symmetric

$$\epsilon_{ik}(\vec{H}) = \epsilon_{ki}(-\vec{H}) \qquad \text{(from generalized principle of symmetry)}$$

No absorption condition requires ϵ_{ik} should be Hermitian, but not that it should be real.

$$\epsilon_{ik} = \epsilon^*_{ki}$$

Let $\epsilon_{ik}(\vec{H}) = \epsilon'_{ik}(\vec{H}) + i\epsilon''_{ik}(\vec{H})$

Real part must be sym $\epsilon'_{ik} = \epsilon'_{ki}$

Im part must be antisym $\epsilon''_{ik} = -\epsilon''_{ki}$

$$\epsilon'_{ik}(\vec{H}) = \epsilon'_{ki}(\vec{H}) = \epsilon'_{ik}(-\vec{H})$$

$$\epsilon''_{ik}(\vec{H}) = -\epsilon''_{ki}(\vec{H}) = -\epsilon''_{ik}(-\vec{H})$$

In a non absorbing medium ϵ'_{ik} is an even function of $\vec{H}$ and ϵ''_{ik} is an odd function of H

Inverse ϵ^{-1}_{ik} has the same symmetry properties

$$\text{Let } \epsilon^{-1}_{ik} = \eta_{ik} = \eta'_{ik} + i\eta''_{ik}$$

Any antisym tensor of rank 2 is axial vector

Let the vector corresponding to tensor η''_{ik} be $\vec{G}$

$$\eta''_{ik} = \epsilon_{ikl} G_l$$

In component form, $\eta''_{xy} = G_z \qquad \eta''_{zx} = G_y \qquad \eta''_{yz} = G_x$

The relation between $\vec{E}$ and $\vec{D}$

$$E_i = \epsilon_{ik}^{-1} D_k = \frac{1}{\epsilon_0}(\eta'_{ik} + i\epsilon_{ikl} G_l) D_k$$

$$E_i = \frac{1}{\epsilon_0}(\eta'_{ik} D_k + i[\vec{D} \times \vec{G}]_i)$$

A medium with such relationship between $\vec{E}$ and $\vec{D}$ is called gyrotropic medium.

Consider now a propagation of a wave in gyrotropic media with no restriction on the magnitude of magnetic fields

$$\text{Substitute } \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{H} = \vec{n} \times \vec{E} \text{ in } -\vec{n} \times \vec{H} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \vec{D}$$

$$\vec{D} = \epsilon_0 [n^2 \vec{E} - \vec{n}(\vec{n} \, . \, \vec{E})]$$

Take propagation along $\vec{k}(\vec{n})$. Transverse component of $\vec{D}$,

$$D_\alpha = \epsilon_0 n^2 E_\alpha$$

$$\Rightarrow E_\alpha = \frac{1}{\epsilon_0}(\epsilon_{\alpha\beta})^{-1} D_\beta$$

$$\Rightarrow D_\alpha - n^2(\epsilon_{\alpha\beta})^{-1} D_\beta = 0$$

$$\text{or} \left(\frac{1}{n^2}\delta_{\alpha\beta} - (\epsilon_{\alpha\beta})^{-1}\right) D_\beta = 0 \qquad \eta_{\alpha\beta} \leftrightarrow \epsilon^{-1}_{\alpha\beta}$$

$$\left(\eta_{\alpha\beta} - \frac{1}{n^2}\delta_{\alpha\beta}\right) D_\beta = 0 \Rightarrow \left(\eta'_{\alpha\beta} + i\eta''_{\alpha\beta} - \frac{1}{n^2}\delta_{\alpha\beta}\right) D_\beta = 0$$

indices α, β are x and y. Propagation along z

x and y are chosen along principal axes of $\eta'_{\alpha\beta}$

Corresponding principal values

$$\frac{1}{n_{01}^2} \text{ and } \frac{1}{n_{02}^2}$$

$$\text{Then, } \left(\eta'_{\alpha\beta} + i\eta''_{\alpha\beta} - \frac{1}{n^2}\delta_{\alpha\beta}\right) D_\beta = 0$$

$$\downarrow$$

$$\left(\frac{1}{n_{01}^2} - \frac{1}{n^2}\right) D_x + iG_z D_y = 0$$

$$-iG_z D_x + \left(\frac{1}{n_{02}^2} - \frac{1}{n^2}\right) D_y = 0$$

Vanishing determinant gives,

$$\left(\frac{1}{n_{01}^2} - \frac{1}{n^2}\right)\left(\frac{1}{n_{02}^2} - \frac{1}{n^2}\right) = G_z^2$$

Roots give two values of n for a given direction $\vec{n}$,

$$\frac{1}{n^2} = \frac{1}{2}\left(\frac{1}{n_{01}^2} + \frac{1}{n_{02}^2}\right) \pm \sqrt{\frac{1}{4}\left(\frac{1}{n_{01}^2} - \frac{1}{n_{02}^2}\right)^2 + G_z^2}$$

$$\Rightarrow \frac{D_y}{D_x} = \frac{i}{G_z}\left\{\frac{1}{2}\left(\frac{1}{n_{01}^2} - \frac{1}{n_{02}^2}\right) \mp \sqrt{\frac{1}{4}\left(\frac{1}{n_{01}^2} - \frac{1}{n_{02}^2}\right)^2 + G_z^2}\right\}$$

Purely imaginary value $\rightarrow$ waves are elliptically polarized

Principal axes are x and y axes

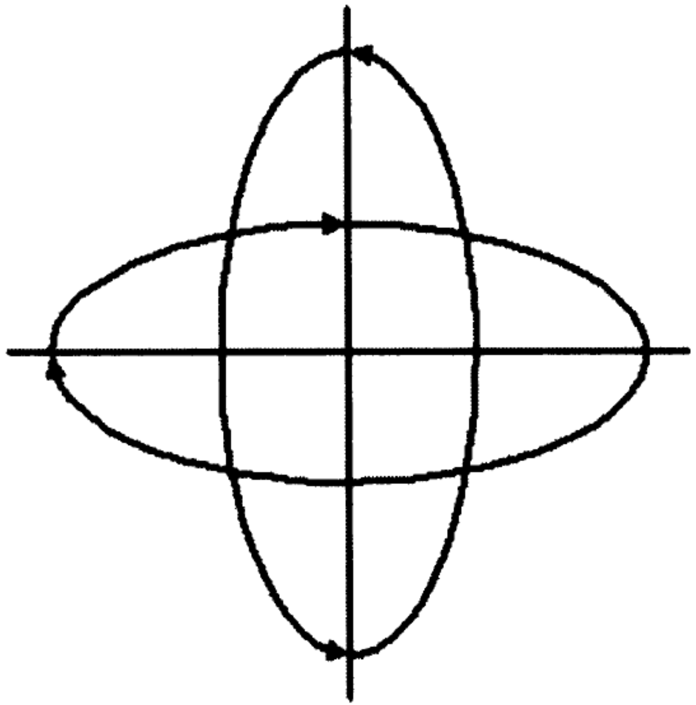
The product of the two values = 1 real

Thus in one wave is $D_y = i\rho D_x$ ρ real - ratio of axes of polarization ellipse

Then the other, $D_y = -i\dfrac{D_x}{\rho}$

$\Rightarrow$ Polarization ellipse of the two waves have the same axis ratio but are 90^0 apart

Direction of rotation opposite



Gyrotropy and magnetic fields

G_i and $\eta'_{ik}-$ functions of magnetic field

$\vec{G}$ is zero in absence of magnetic field. Thus for weak field,

$$G_i = f_{ik} H_k \qquad f_{ik}\text{- tensor of rank 2}$$

$$f_{ik} \rightarrow \text{In general not symmetrical}$$

Components of antisymmetric tensor η''_{ik} must be odd functions of $\vec{H}$

For arbitrary direction of propagation- magnetic field has very little effect.

Effects are larger near optic axes

Two values of n are equal in absence of the field when wave vector is along one of these axes

Magneto optic effect in isotropic bodies and in cubic crystals- interesting

$$\eta'_{ik} = \epsilon_r^{-1} \delta_{ik}$$

ϵ- dielectric constant of isotropic material in absence of $\vec{H}$

$\vec{E} \leftrightarrow \vec{D}$ relation

$$\vec{E} = \frac{1}{\epsilon_0}\left(\frac{1}{\epsilon_r}\vec{D} + i\vec{D} \times \vec{G}\right)$$

where $\vec{G} = \frac{-\vec{g}}{(\epsilon_r)^2}$

$$\vec{D} = \epsilon_0(\epsilon_r \vec{E} + i\vec{E} \times \vec{g})$$

$$\vec{g} = f\vec{H} \qquad f \text{ - scalar constant}$$

$$n_{01} = n_{02} = n_0 = \sqrt{\epsilon_r}$$

$$\left(\frac{1}{n_{01}^2} - \frac{1}{n^2}\right)\left(\frac{1}{n_{02}^2} - \frac{1}{n^2}\right) = G_z^2$$

$$\text{Hence, } \frac{1}{n^2} = - \pm G_z + \frac{1}{n_0^2}$$

To same accuracy $\quad n_{\mp}^2 = n_0^2 \pm n_0^4 G_z = n_0^2 \mp g_z$

$$n_{\mp}^2 = \frac{n_0^2}{1 \mp G_z n_0^2}$$

Since z axis is in $\vec{n}$ direction,

$$\left(\vec{n} \pm \frac{1}{2n_0}\vec{g}\right)^2 = n_0^2$$

$\Rightarrow$ Wave vector surface $\rightarrow$ two spheres of radius n_0

with separated centers by $\pm\frac{g}{2n_0}$ from origin in the direction of $\vec{g}$ or $\vec{G}$

Different polarization correspond to each of the two waves,

$$D_x = \mp i D_y \qquad \text{(RCP and LCP)}$$

Two circularly polarized waves have different wave vector magnitudes

$$k_\pm = \frac{\omega}{c} n_\pm$$

Linear polarization → sum of RCP + LCP

$$D_x = \frac{1}{2}[\exp(ik_+z) + \exp(ik_-z)] \qquad D_y = \frac{1}{2}[i(-\exp(ik_+z) + \exp(ik_-z))]$$

Introduce $k = \dfrac{k_+ + k_-}{2}$, $\kappa = \dfrac{k_+ - k_-}{2}$

$$D_x = \frac{1}{2}e^{ikz}\left[e^{i\kappa z} + e^{-i\kappa z}\right] = e^{ikz}\cos\kappa z$$

$$D_y = \frac{1}{2}ie^{ikz}\left[-e^{i\kappa z} + e^{-i\kappa z}\right] = e^{ikz}\sin\kappa z$$

After exiting from the slab,

$$\frac{D_y}{D_x} = \tan\kappa l = \tan\frac{l\omega g}{2cn_0} \longrightarrow \text{Real}$$

Since the ratio is real, wave remains linearly polarized.

Direction of polarization changes

$\Rightarrow$ Faraday's effect

Angle through which plane of polarization is rotated $\sim$ path traversed

Angle / unit length in the direction of the wave vector is $\left(\frac{\omega g}{2cn_0}\right)\cos\theta$

$\theta-$ angle between $\vec{n}$ and $\vec{g}$

$\theta = \frac{\pi}{2} \rightarrow$ one needs to include quadratic in $\vec{H}$ terms

$\Rightarrow$ Cotton-Mouton effect