Condensed Matter Physics II

Midterm Exam

April 24, 2020

Total: 100 Marks

1. Consider the tensor product of two spin-1/2 systems, described by Pauli spin operators σ_1^x, σ_1^y, σ_1^z and σ_2^x, σ_2^y, σ_2^z respectively. The Heisenberg Hamiltonian can be written in the tensor product space as

$$\mathcal{H} = -J\vec{\sigma}_1.\vec{\sigma}_2 = -J\left(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + \sigma_1^z\sigma_2^z\right), \tag{1}$$

where J represents the strength of interaction between spins.

   (a) Express the Hamiltonian as a matrix in terms of basis states constructed from the tensor product of $|\uparrow\rangle_1$, $|\downarrow\rangle_1$ and $|\uparrow\rangle_2$, $|\downarrow\rangle_2$, the eigenstates of σ_1^z and σ_2^z respectively.

   (b) Find the eigenvalues and eigenvectors of this Hamiltonian.

   (20 Marks)

2. Consider a system of three particles in one dimension, with one particle in a state $\phi_a(x)$ another in state $\phi_b(x)$ and another in $\phi_c(x)$.

   (a) Express the many body wavefunction of this system for the case when the particles are (i) bosons and (ii) fermions.

   (b) In each case, compute the expectation value of

$$\mathcal{O} = (x_1 - x_2 + x_3)^2. \tag{2}$$

   (30 Marks)

3. Consider the two site Hubbard model

$$H = -t\sum_{\sigma}\left(c_{2\sigma}^{\dagger}c_{1\sigma} + c_{1\sigma}^{\dagger}c_{2\sigma}\right) + U\sum_{i=1,2} n_{i\uparrow}n_{i\downarrow}. \tag{3}$$

   Represent the Fock space Hamiltonian for the case when $N = N_\uparrow + N_\downarrow = 3$.

   (20 Marks)

4. Consider the system in Question 1 in an external magnetic field h, described by the Hamiltonian

$$\mathcal{H} = -J\left(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + \sigma_1^z\sigma_2^z\right) + h\left(\sigma_1^z + \sigma_2^z\right). \tag{4}$$

   (a) Compute the partition function of this system, given by

$$\mathcal{Z} = \mathrm{Tr}\left(\exp(-\beta\mathcal{H})\right). \tag{5}$$

   (b) Compute the average value of the magnetization $m = \sigma_1 + \sigma_2$. (Note: this is a tensor product operator).

   (30 Marks)