Condensed Matter Physics

Final Exam

February 6, 2021

Total: 100 Marks

1. Consider a linear chain representing a Bravais lattice composed of N units cells of size a, with a two-atom basis. All atoms are constrained to move only along the chain. The two inequivalent atoms, A and B, have masses $m_A = 2m$ and $m_B = m$, respectively. B atoms are connected to nearest-neighbors A atoms by inequivalent springs, whose elastic constant are $K_1 = 2K$ and $K_2 = K$, respectively (see Figure). Indicate with u_n^A and u_n^B the displacement of A and B atoms in the n-th unit cell, with respect to their equilibrium positions. Adopt Born-von Karman periodic boundary conditions. To assign numerical values, consider $m = 92 \times 10^{-27}$ kg, $a = 0.5$ nm, $K = 4.5$ kg/s^2.

   (a) Assuming traveling-wave solutions

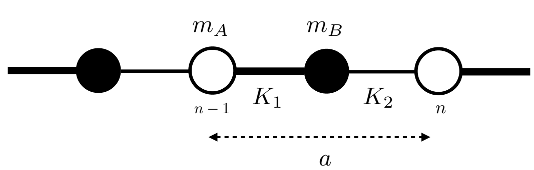
   $$u_n^A = A \exp\left[i(qna - \omega t)\right],$$
   $$u_n^B = B \exp\left[i(qna - \omega t)\right].$$

   where q is the wave vector, determine the dispersion of the acoustic and optical phonon branches, $\omega_a(q)$ and $\omega_o(q)$.

   (b) Verify that, for small q, $\omega_a(q) \approx c_s|q|$ and determine the numerical value of the velocity of sound c_s.

   (c) Adopting a Debye model for the acoustic branch, $\omega_a = c_s|q|$, and an Einstein model for the optical branch, with $\omega_E = \omega_o(q = 0)$, determine the low-temperature and high-temperature asymptotic expressions for the specific heat c_V of the lattice. Provide the value of c_V at $T = 1$K. Consider that $\int_0^\infty \frac{x}{\exp(x) - 1} = \frac{\pi^2}{6}$.

   (30 Marks)



2. Electrons in a piece of gold metal can be assumed to behave like an ideal Fermi gas and follow the Sommerfeld theory of metals. Gold metal in the solid state has a mass density of 19.30 g/cm^3.

   (a) Compute the mean free path of electrons in the material. Estimate the conductivity using either Drude or Sommerfeld theory (make reasonable assumptions for the values).

   (b) Assume that each gold atom donates one electron to the Fermi gas. Assume the system is in the ground-state ($T = 0$ K). Compute the Fermi speed, Fermi energy (in eV) and the Fermi temperature.

   (20 Marks)

3. Nowadays it is experimentally possible to confine electrons in thin layers forming two-dimensional (2D) systems or in thin wires forming one-dimensional (1D) systems. Consider a model of non-interacting electrons (i.e. ideal gas of spin-1/2 fermions) in 2D and 1D. Assume the electrons to be confined in 2D within a square of area $A = L^2$ and in 1D within a line of length L.

   (a) Express the Fermi wave vector (k_F), the Fermi energy (E_F), and the total energy per unit of area of the system as a function of the electron density ($n_{2D} = N/A$ or $n_{1D} = N/L$).

   (b) Calculate the density of states of the system $g_{\text{2D}}(\epsilon)$ and $g_{\text{1D}}(\epsilon)$.

   (20 Marks)

4. Electrons in a crystal are subject to the one-dimensional periodic potential

$$V(x) = V_0 + A\cos\left(\frac{4\pi x}{a}\right) + B\cos\left(\frac{6\pi x}{a}\right).$$

   (a) Under what conditions will the nearly free-electron approximation work? Assuming that the conditions are satisfied, sketch the four lowest energy bands. Number the energy bands starting from one at the lowest band.

   (b) Applying the nearly free-electron approximation at the four lowest energy bands, calculate (to the first-order) the energy gaps at $k = 0$ and $k = \pi/a$.

   (c) Answer the same questions in (b) assuming a one-dimensional potential $V(x)$ composed by a periodic sequence of δ-like potential barriers

$$V(x) = \sum_n aV_0\delta(x - na)$$

   where a is the lattice constant.

   (30 Marks)

Useful Constants

Atomic Weight of Gold: 197 a.u.
Planck's constant: $h = 6.62607 \times 10^{-34}$ m^2 kg s^{-1}
Boltzmann constant: $k_B = 1.38064 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}