Condensed Matter Physics

Assignment II

Due: January 10, 2021

1. **Fermi gases in astrophysics**

   (a) Given $M_\odot = 2 \times 10^{33}$ g for the mass of the Sun, estimate the number of electrons in the Sun. In a white dwarf star this number of electrons may be ionized and contained in a sphere of radius 2×10^9 cm; find the Fermi energy of the electrons in electron volts.

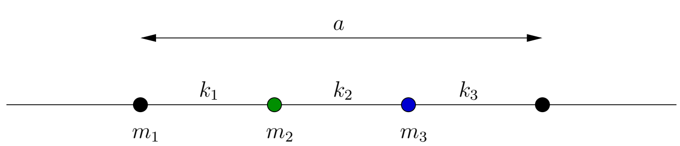
   (b) The energy of an electron in the relativistic limit $\epsilon \gg mc^2$ is related to the wavevector as $\epsilon \approx pc = \hbar kc$ Show that the Fermi energy in this limit is $\epsilon_F \approx \hbar c(N/V)^{1/3}$, roughly.

   (c) If the above number of electrons were contained within a pulsar of radius 10 km, show that the Fermi energy would be 10^8 eV. This value explains why pulsars are believed to be composed largely of neutrons rather than of protons and electrons, for the energy release in the reaction $n \to p + e^-$ is only 0.8×10^6 eV, which is not large enough to enable many electrons to form a Fermi sea. The neutron decay proceeds only until the electron concentration builds up enough to create a Fermi level of 0.8×10^6 eV, at which point the neutron, proton, and electron concentrations are in equilibrium.

   (25 Marks)

2. **Phonons in a Triatomic Chain**

   Consider a mass-and-spring model with three different masses and three different springs per unit cell as shown in this diagram.



   Assume that the masses move only in one dimension.

   (a) At $k = 0$ how many optical modes are there? Calculate the energies of these modes.

   (b) If all the masses are the same and $k_1 = k_2$, determine the frequencies of all three modes at the zone boundary $k = \pi/a$ (you should be able to guess one root of the resulting cubic equation).

   (c) Similarly, if all three spring constants are the same, and $m_1 = m_2$, determine the frequencies of all three modes at the zone boundary $k = \pi/a$.

   (25 Marks)

3. **Diatomic Einstein Solid**

   Consider a three-dimensional simple harmonic oscillator with mass m and spring constant k (i.e., the mass is attracted to the origin with the same spring constant in all three directions). The Hamiltonian is given by

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{k}{2}\mathbf{x}^2. \tag{1}$$

Note: $\mathbf{p}$ and $\mathbf{x}$ are three dimensional vectors.

(a) Calculate the classical partition function

$$Z = \int \frac{\mathbf{dp}}{2\pi\hbar} \int \mathbf{dx} \exp\left(-\beta \mathcal{H}(\mathbf{p}, \mathbf{x})\right). \tag{2}$$

(b) Next, consider the same Hamiltonian quantum mechanically. Calculate the quantum partition function

$$Z = \sum_j \exp(-\beta E_j). \tag{3}$$

Explain the relationship with Bose statistics.

(c) Next, consider a solid made up of diatomic molecules, modeled as two particles in three dimensions connected to each other with a spring, both at the bottom of a harmonic well. Here

$$\mathcal{H} = \frac{\mathbf{p_1}^2}{2m_1} + \frac{\mathbf{p_2}^2}{2m_2} + \frac{k}{2}\mathbf{x_1}^2 + \frac{k}{2}\mathbf{x_2}^2 + \frac{K}{2}(\mathbf{x_1} - \mathbf{x_2})^2, \tag{4}$$

where k is the spring constant holding both particles in the bottom of the well, and K is the spring constant holding the two particles together. Assume that the two particles are distinguishable atoms.

(d) Analogous to (a) calculate the classical partition function and show that the heat capacity is $3k_B$ per particle.

(e) Analogous to (b) calculate the quantum partition function and find an expression for the heat capacity. Sketch the heat capacity as a function of temperature if $K \gg k$.

(30 Marks)

4. **van Hove Singularities**

(a) In a linear harmonic chain with only nearest-neighbor interactions. The normal-mode dispersion relation has the form $\omega(k) = \omega_0 |\sin(ka/2)|$, where the constant ω_0 is the maximum frequency (assumed when k is on the Brillouin zone boundary). Show that the density of normal modes in this case is given by

$$g(\omega) = \frac{2}{\pi a \sqrt{\omega_0^2 - \omega^2}}. \tag{5}$$

The singularity at $\omega = \omega_0$ is a van Hove singularity.

(b) In three dimensions the van Hove singularities are infinities not in the normal mode density itself: but in its derivative. Show that the normal modes in the neighborhood of a maximum of $\omega(\mathbf{k})$ for example, lead to a term in the normal-mode density that varies as $(\omega_0 - \omega)^{1/2}$.

(20 Marks)