Condensed Matter Physics

Assignment I

Due: October 10, 2019

1. **Bose-Einstein Distribution**
   Derive the occupation probability of energy levels for a collection of bosons

$$f(E) = \frac{1}{\exp\left(\frac{E-\mu}{k_B T}\right) - 1}, \tag{1}$$

   where T is the temperature, μ is the chemical potential and k_B is the Boltzmann constant.

   (20 Marks)

2. **Poisson Distribution**
   In the Drude model the probability of an electron suffering a collision in any infinitesimal interval dt is dt/τ.

   (a) Show that an electron picked at random at a given moment had no collision during the preceding t seconds with probability $\exp(-t/\tau)$. Show that it will have no collision during the next t seconds with the same probability.

   (b) Show that the probability that the time interval between two successive collisions of an electron falls in the range between t and $t + dt$ is $(dt/\tau)\exp(-t/\tau)$.

   (c) Show as a consequence of (a) that at any moment the mean time back to the last collision (or up to the next collision) averaged over all electrons is τ.

   (d) Show as a consequence of (b) that the mean time between successive collisions of an electron is τ.

   (e) Derive the probability distribution for T, the time between the last and next collision.

   (25 Marks)

3. **Joule Heating**
   Consider a metal at uniform temperature in a static uniform electric field E. An electron experiences a collision, and then, after a time t, a second collision. In the Drude model, energy is not conserved in collisions, for the mean speed of an electron emerging from a collision does not depend on the energy that the electron acquired from the field since the time of the preceding collision.

   (a) Show that the average energy lost to the ions in the second of two collisions separated by a time t is $(eEt)^2/2m$ (the average is over all directions in which the electron emerged from the first collision).

   (b) Show that the average energy loss to the ions per electron per collision is $(eEt)^2/m$, and hence that the average loss per cubic centimeter per second is $(ne^2\tau/m)E^2 = \sigma E^2$. Deduce that the power loss in a wire of length L and cross section A is I^2R, where I is the current flowing and R is the resistance of the wire.

   (25 Marks)

4. **Diatomic Einstein Solid**
Consider a three-dimensional simple harmonic oscillator with mass m and spring constant k (i.e., the mass is attracted to the origin with the same spring constant in all three directions). The Hamiltonian is given by

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{k}{2}\mathbf{x}^2. \tag{2}$$

Note: $\mathbf{p}$ and $\mathbf{x}$ are three dimensional vectors.

(a) Calculate the classical partition function

$$Z = \int \frac{\mathbf{dp}}{2\pi\hbar} \int \mathbf{dx} \exp\left(-\beta\mathcal{H}(\mathbf{p},\mathbf{x})\right). \tag{3}$$

(b) Next, consider the same Hamiltonian quantum mechanically. Calculate the quantum partition function

$$Z = \sum_j \exp(-\beta E_j). \tag{4}$$

Explain the relationship with Bose statistics.

(c) Next, consider a solid made up of diatomic molecules, modeled as two particles in three dimensions connected to each other with a spring, both at the bottom of a harmonic well. Here

$$\mathcal{H} = \frac{\mathbf{p_1}^2}{2m_1} + \frac{\mathbf{p_2}^2}{2m_2} + \frac{k}{2}\mathbf{x_1}^2 + \frac{k}{2}\mathbf{x_2}^2 + \frac{K}{2}(\mathbf{x_1} - \mathbf{x_2})^2, \tag{5}$$

where k is the spring constant holding both particles in the bottom of the well, and K is the spring constant holding the two particles together. Assume that the two particles are distinguishable atoms.

(d) Analogous to (a) calculate the classical partition function and show that the heat capacity is $3k_B$ per particle.

(e) Analogous to (b) calculate the quantum partition function and find an expression for the heat capacity. Sketch the heat capacity as a function of temperature if $K \gg k$.

(30 Marks)