Spatial Extent of Branching Brownian Motion

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Presentation Outline

1. Introduction
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4. Results
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Extreme Value Statistics

- Extreme value statistics has been growing in prominence.
- In many real world examples the extreme value is not independent of the rest of the set and there are **strong correlations between near-extreme values**.
- Examples include **extreme temperatures** as part of heat or cold waves, **earthquakes and financial crashes** where extreme fluctuations are accompanied by foreshocks and aftershocks.
- Particularly important in **disordered systems** where energy levels near the ground state become important at low but finite temperature.
- Although EVS of independent identically distributed (i.i.d.) variables are fully understood, **very few analytical results for strongly correlated random variables**.

Branching Brownian Motion

At each time step $[t, t + \Delta t]$ the particle can:

- **A)** die with probability $a\Delta t$
- **B)** split into two independent particles with probability $b\Delta t$
- **C)** diffuse by a distance $\Delta x = \eta(t)\Delta t$, with probability $1 - (a + b)\Delta t$.

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t_1)\eta(t_2) \rangle = 2D\delta(t_1 - t_2) \tag{1}$$

Branching Brownian Motion

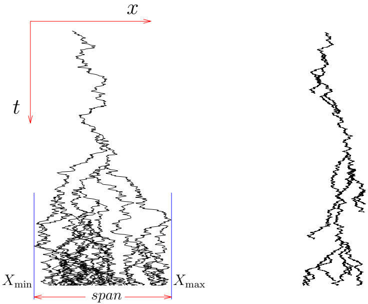
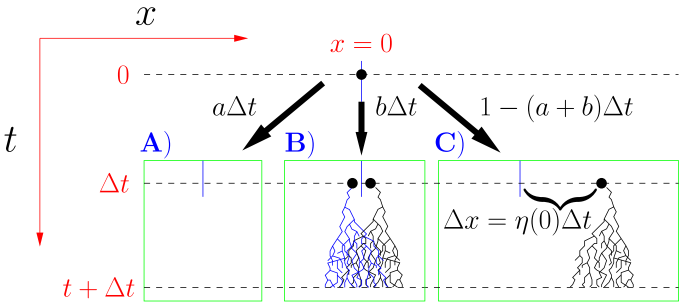


Figure: A realization of the dynamics of branching Brownian motion with death (left) in the supercritical regime ($b > a$) and (right) in the critical regime ($b = a$). The particles are numbered sequentially from right to left as shown in the inset.

The Backward Fokker-Planck Approach

- We look at the contribution from the **first time step** $[0, \Delta t]$ to the final time step $t + \Delta t$



Number of Particles in the system

- $P(n,t) =$ Probability there are exactly n particles at time t.

- Using the Backward Fokker-Planck approach

$$\begin{aligned} P(n,t+\Delta t) &= [1-(a+b)\Delta t]P(n,t) + \\ & b\Delta t \sum_{m=0}^{n} P(m,t)P(n-m,t) + a\Delta t\, \delta_{n,0}\ . \end{aligned} \quad (2)$$

- In the $\Delta t \to 0$ we have

$$\frac{\partial P(n,t)}{\partial t} = -(a+b)P(n,t) + b\sum_{m=0}^{n} P(m,t)P(n-m,t) + a\,\delta_{n,0}\ . \quad (3)$$

- We can solve this using standard **generating functions**.

Number of Particles in the system

- The solutions are

$$P(0,t) = \frac{a(e^{bt} - e^{at})}{be^{bt} - ae^{at}} \ , \qquad P(n \geq 1, t) = (b-a)^2 e^{(a+b)t} \frac{b^{n-1}(e^{bt} - e^{at})^{n-1}}{(be^{bt} - ae^{at})^{n+1}} \ . \tag{4}$$

- In the critical regime $(b = d)$ this reduces to

$$P(0,t) = \frac{bt}{1+bt}, \qquad P(n \geq 1, t) = \frac{(bt)^{n-1}}{(1+bt)^{n+1}} \ . \tag{5}$$

- The average number of particles is

$$\langle N(t) \rangle = e^{(b-a)t}. \tag{6}$$

The Maximum of BBM

- $Q(X, t)$ = Probability that $X_{max} \leq X$ at time t
- PDF of X_{max}:

$$P_{\mathrm{marg}}(X, t) = \frac{\partial}{\partial X} Q(X, t). \tag{7}$$

- The initial condition is

$$Q(X, 0) = \theta(X) \tag{8}$$

- The boundary conditions are

$$Q(X, t) = \begin{cases} 1 & \text{for} \quad X \to \infty \\ 0 & \text{for} \quad X < 0. \end{cases} \tag{9}$$

The Maximum of BBM (Cont.)

- Using the backward Fokker-Planck approach, we have

$$\begin{aligned} Q(X, t+\Delta t) &= (1-(a+b)\Delta t)\,\langle Q(X-\eta(0)\Delta t, t)\rangle_{\eta(0)} \\ &+ b\Delta t Q^2(X,t) + a\Delta t \;. \end{aligned} \tag{10}$$

- In the $\Delta t \to 0$ we have

$$\frac{\partial Q(X,t)}{\partial t} = D\frac{\partial^2 Q(X,t)}{\partial X^2} - (a+b)Q(X,t) + bQ^2(X,t) + a \tag{11}$$

- In terms of $R(X,t) = 1 - Q(X,t)$:

$$\frac{\partial R(X,t)}{\partial t} = D\frac{\partial^2 R(X,t)}{\partial X^2} + (b-a)\; R(X,t) - b\; R^2(X,t), \tag{12}$$

- Non-linear equation with no known general solution ($a = 0$ is the famous **Fisher-Kolmogorov-Pertrovski-Piscounov Equation**).

Dimensionless Variables

It is natural to consider the evolution equations in terms of dimensionless variables as follows

$$x = \frac{X}{\sqrt{D/b}}, \qquad \left(y = \frac{Y}{\sqrt{D/b}}, \quad s = \frac{\zeta}{\sqrt{D/b}} \right),$$
$$\tau = bt,$$
$$\Delta = \frac{a}{b} - 1. \tag{13}$$

In terms of these dimensionless variables Eq. (12) takes the simpler form

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial^2 R(x,t)}{\partial x^2} - \Delta R(x,y,\tau) - R^2(x,y,\tau). \tag{14}$$

For $\Delta > 0$: expected to approach a **stationary limit** as $\tau \to \infty$:

$$\mathcal{R}(x) = R(x,y,\tau \to \infty). \tag{15}$$

The Maximum of BBM: Stationary Critical Solution

In the critical case $\Delta = 0$ (S. Sawyer and J. Fleischman, Proc. Natl. Acad. Sci. USA **76**(2), 87 (1979)):

$$\mathcal{R}(x) = \frac{1}{\left(1 + \frac{x}{\sqrt{6}}\right)^2} \,. \tag{16}$$

Consequently, $p_{\mathrm{marg}}(x) = -d\mathcal{R}(x)/dx$ has the asymptotic behaviors

$$p_{\mathrm{marg}}(x) \sim \begin{cases} p_{\mathrm{marg}}(0) = \sqrt{\frac{2}{3}} \,, \; x \to 0 \\ \\ \frac{12}{x^3} \,, \; x \to \infty \,. \end{cases} \tag{17}$$

The Maximum of BBM: Stationary subcritical Solution

In the subcritical case $\Delta > 0$ (S. Sawyer and J. Fleischman, Proc. Natl. Acad. Sci. USA **76**(2), 87 (1979)):

$$\mathcal{R}(x) = \frac{3\Delta}{2}\mathrm{csch}^2\left(\frac{\sqrt{\Delta}}{2}x + \sinh^{-1}\sqrt{\frac{3\Delta}{2}}\right). \tag{18}$$

Consequently, $p_{\mathrm{marg}}(x) = -d\mathcal{R}(x)/dx$ has the asymptotic behaviors

$$p_{\mathrm{marg}}(x) \sim \begin{cases} p_{\mathrm{marg}}(0) = \sqrt{\frac{2}{3} + \Delta}\,,\ x \to 0 \\ \\ 6\Delta^{\frac{3}{2}} e^{-2\sinh^{-1}\sqrt{\frac{3\Delta}{2}}} \exp\left(-\sqrt{\Delta}\,x\right)\,,\ x \to \infty\,. \end{cases} \tag{19}$$

Joint Distribution of the Maximum and Minimum

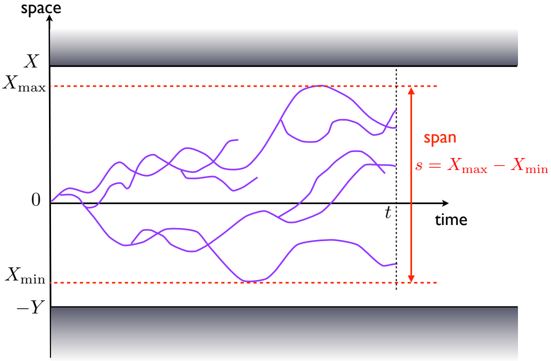


Figure: Schematic representation of a trajectory of the BBM confined in the box $[-Y, X]$. Note that X_{max} and X_{min} denote respectively the maximum and the minimum of the process *up to time t*. The process starts with a single particle at the origin at time $t = 0$ and hence $X_{\text{max}} \geq 0$ while $X_{\text{min}} \leq 0$.

Joint Distribution of the Maximum and Minimum

- $Q(X, Y, t) =$ Joint probability that $X_{max} \leq X$ AND $X_{min} \geq -Y$ at time t
- Using the backward Fokker-Planck approach, we have

$$\begin{aligned} Q(X, Y, t + \Delta t) &= b\Delta t \; Q^2(X, Y, t) + a\Delta t \\ &+ (1 - (b + a)\Delta t) \langle Q(X - \Delta x, Y + \Delta x, t) \rangle_{\eta(0)}, \end{aligned} \tag{20}$$

- In the limit $\Delta t \to 0$, we arrive at the exact BFP evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} Q(X, Y, t) = D \left(\frac{\partial}{\partial X} - \frac{\partial}{\partial Y} \right)^2 Q(X, Y, t) + a \\ -(b + a) \; Q(X, Y, t) + b \; Q^2(X, Y, t). \end{aligned} \tag{21}$$

- The initial condition is

$$Q(X,Y,0) = \Theta(X)\Theta(Y)\ , \tag{22}$$

- The boundary conditions are

$$Q(X,Y,t) = \begin{cases} 0 & \text{for} \qquad X < 0, \\ 0 & \text{for} \qquad Y < 0\ . \end{cases} \tag{23}$$

In terms of the complimentary probability $R(X,Y,t) = 1 - Q(X,Y,t)$, we have

$$\frac{\partial R(X,Y,t)}{\partial t} = D\left(\frac{\partial}{\partial X} - \frac{\partial}{\partial Y}\right)^2 R(X,Y,t) + (b-a)\, R(X,Y,t) - b\, R^2(X,Y,t), \tag{24}$$

with the initial conditions

$$R(X,Y,0) = \Theta(-X)\Theta(-Y). \tag{25}$$

and the boundary conditions

$$R(X,Y,t) = \begin{cases} 1 & \text{for} \quad X < 0, \\ 1 & \text{for} \quad Y < 0. \end{cases} \tag{26}$$

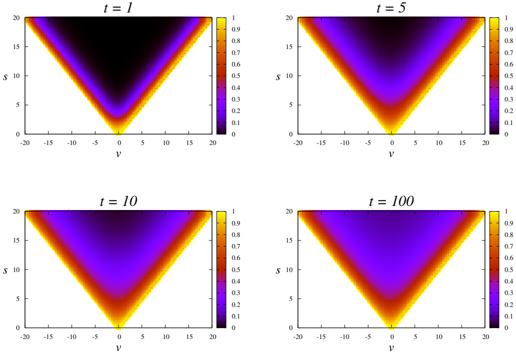


Figure: The solutions $R(s, v, t)$ at different times obtained by numerical integration, with $dx = 0.1, dt = 0.001$, $D = 1$ and $b = 0.5$. We find that at large times this converges to a stationary bivariate function $\mathcal{R}(s, v)$.

Distribution of the Span

- The distribution of the span $s = X_{max} - X_{min}$ is given by

$$P(s,t) = \int_0^\infty \int_0^\infty dXdY\delta(X+Y-s)P(X,Y,t). \qquad (27)$$

- In terms of the dimensionless variables

$$P(\zeta,\tau) = \int_0^\infty \int_0^\infty dxdy\delta(x+y-\zeta)P(x,y,\tau). \qquad (28)$$

- $$p_{\rm uncorr}(\zeta) = \int_0^\zeta p_{\rm marg}(x)p_{\rm marg}(\zeta - x)dx\ . \qquad (29)$$

Change of Variables

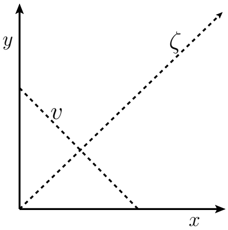


Figure: The change of variables $\{x, y\} \to \{\zeta, v\}$.

Next, it is convenient to make a change of variables

$$\zeta = x + y,$$
$$v = x - y, \quad (30)$$

with $\zeta \in [0, \infty)$ and $v \in [-\zeta, \zeta]$. ζ **represents the dimensionless span of the process**.

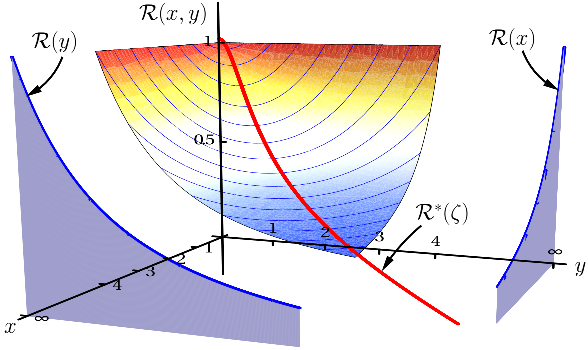


Figure: Graph depicting the stationary joint cumulative probability $\mathcal{R}(x, y)$. The limiting distributions correspond to the marginal probabilities of the maximum and minimum $\mathcal{R}(x) = \mathcal{R}(x, y \to \infty)$ and $\mathcal{R}(y) = \mathcal{R}(x \to \infty, y)$ respectively.

- In terms of these new variables Eq. (24) becomes

$$4\left(\frac{\partial}{\partial v}\right)^2 \mathcal{R}(\zeta, v) - \Delta\mathcal{R}(\zeta, v) - \mathcal{R}^2(\zeta, v) = 0 \ , \tag{31}$$

valid in the regime $v \in [-\zeta, +\zeta]$ and $\zeta \in [0, +\infty)$.

- For a fixed ζ, Monotonically decreasing $v \in [-\zeta, 0]$, Monotonically increasing $v \in [0, \zeta]$.
- Assuming analyticity around the minimum at $v = 0$ gives the condition

$$\left.\frac{\partial\mathcal{R}(\zeta, v)}{\partial v}\right|_{v=0} = 0. \tag{32}$$

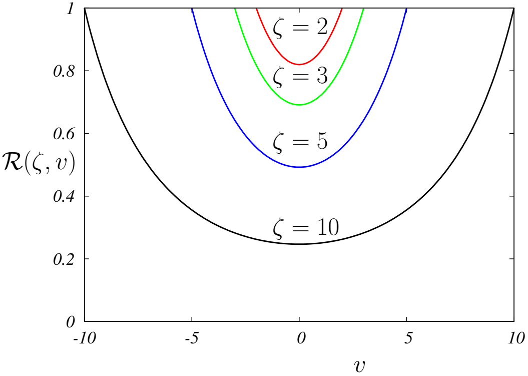


Figure: $\mathcal{R}(\zeta, v)$ as a function of $v \in [-\zeta, +\zeta]$ for different values of ζ. For fixed ζ, $\mathcal{R}(\zeta, v)$ is a smooth non-monotonic function, symmetric around $v = 0$ in $-\zeta \leq v \leq +\zeta$, and has a minimum at $v = 0$.

- Fortunately, Eq. (31) can be integrated with respect to v upon multiplying by a factor $2\frac{\partial \mathcal{R}(\zeta,v)}{\partial v}$, yielding

$$\left(\frac{\partial \mathcal{R}(\zeta,v)}{\partial v}\right)^2 = \frac{\Delta}{4}\,\mathcal{R}^2(\zeta,v) + \frac{1}{6}\,\mathcal{R}^3(\zeta,v) + \kappa(\zeta), \qquad (33)$$

where $\kappa(\zeta)$ is a yet unknown integration constant.

- To fix $\kappa(\zeta)$, we use the condition in Eq. (32) and arrive at

$$\left(\frac{\partial \mathcal{R}(\zeta,v)}{\partial v}\right)^2 = \frac{\Delta}{4}\,\left(\mathcal{R}^2(\zeta,v) - \mathcal{R}^2(\zeta,0)\right) + \frac{1}{6}\,\left(\mathcal{R}^3(\zeta,v) - \mathcal{R}^3(\zeta,0)\right)$$

(3

- This equation can be conveniently expressed as

$$\frac{1}{\sqrt{\mathcal{R}(\zeta,0)}}\mathcal{G}\left(\frac{3\Delta/2}{\mathcal{R}(\zeta,0)},\frac{\mathcal{R}(\zeta,v)}{\mathcal{R}(\zeta,0)}\right)=\frac{v}{\sqrt{6}}, \tag{35}$$

where the bivariate function $\mathcal{G}$ is defined by the integral

$$\mathcal{G}(\gamma,z)=\int_1^z \frac{dx}{\sqrt{(x^3-1)+\gamma\,(x^2-1)}}\,, \tag{36}$$

- The above function $\mathcal{G}(\gamma,z)$ can then be expressed as

$$\mathcal{G}(\gamma,z)=\frac{1}{(3+2\gamma)^{1/4}}\mathbf{F}\left[\tan^{-1}\sqrt{\frac{z-1}{\sqrt{3+2\gamma}}}\,,\ \frac{2\sqrt{3+2\gamma}-(3+\gamma)}{4\sqrt{3+2\gamma}}\right], \tag{37}$$

where $z\geq 1$, $\gamma\geq 0$ and $\mathbf{F}$ is the elliptic integral of the first kind.

$$\mathbf{F}(\phi,k)=\int_0^\phi \frac{d\theta}{\sqrt{1-k\sin^2\theta}} \tag{38}$$

- Next, inserting the boundary condition $\mathcal{R}(\zeta, \pm\zeta) = 1$ in the above equation we have

$$\frac{1}{\sqrt{\mathcal{R}(\zeta,0)}}\mathcal{G}\left(\frac{3\Delta/2}{\mathcal{R}(\zeta,0)}, \frac{1}{\mathcal{R}(\zeta,0)}\right) = \frac{\zeta}{\sqrt{6}}. \tag{39}$$

- This is an **implicit equation** for $\mathcal{R}(\zeta, 0)$, the solution of which can then be injected in Eq. (35) to solve for $\mathcal{R}(\zeta, v)$ for all ζ and v.

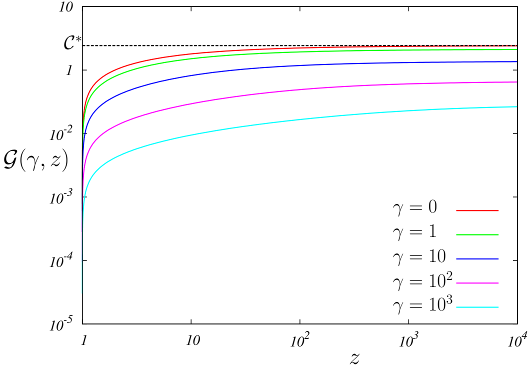


Figure: The function $\mathcal{G}(\gamma, z)$ for different values of γ. For large z, $\mathcal{G}(\gamma, z)$ saturates to a γ dependent constant $\mathcal{C}(\gamma)$. The case $\gamma = 0$ corresponds to the function $\mathcal{G}(0, z)$ analyzed in the critical case. The limiting behaviors are $\mathcal{G}(0, z) \to 0$ as $z \to 1$ and $\mathcal{G}(0, z) \to \mathcal{C}^* = \frac{\sqrt{\pi}}{3} \frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})} \approx 2.4286$ as $z \to \infty$.

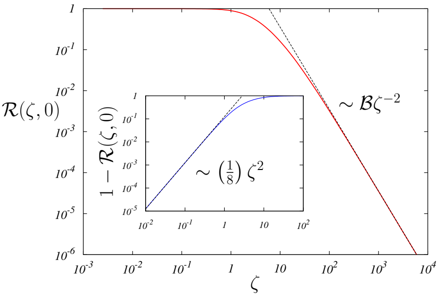


Figure: The function $\mathcal{R}(\zeta, 0)$ versus ζ in the critical regime, showing the limiting behaviors $\mathcal{R}(\zeta, 0) \to 1$ as $\zeta \to 0$ and $\mathcal{R}(\zeta, 0) \to \frac{\mathcal{B}}{\zeta^2}$ as $\zeta \to \infty$ (dashed line). $\mathcal{B} \approx 35.3901$. **Inset**: Plot of $1 - \mathcal{R}(\zeta, 0)$ showing the limiting behavior $1 - \mathcal{R}(\zeta, 0) \sim \left(\frac{1}{8}\right) \zeta^2$ as $\zeta \to 0$ (dashed line).

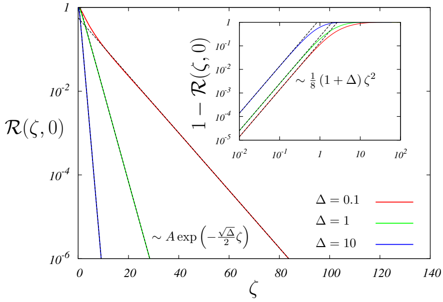


Figure: Asymptotic behavior of $\mathcal{R}(\zeta, 0)$ in the subcritical regime. The plot shows $\mathcal{R}(\zeta, 0)$ for different values of Δ. The dashed lines representing the asymptotic exponential behavior $\mathcal{R}(\zeta, 0) \sim A \exp\left(-\frac{\sqrt{\Delta}}{2}\zeta\right)$ as $\zeta \to \infty$ are indistinguishable from the theoretically obtained curves as they match exactly. **Inset**: Plot of $1 - \mathcal{R}(\zeta, 0)$ showing the limiting behavior $1 - \mathcal{R}(\zeta, 0) \sim \frac{1}{8}(1+\Delta)\zeta^2$ as $\zeta \to 0$ (dashed lines).

$\zeta \to 0$ Asymptotics

- To leading order in ζ we have

$$\mathcal{R}(\zeta,0) = 1 - \frac{1}{8}(1+\Delta)\zeta^2 + \mathcal{O}(\zeta^4). \tag{40}$$

- Therefore

$$\mathcal{R}(\zeta,v) = 1 - \frac{1}{8}\left(1+\Delta\right)\left(\zeta^2 - v^2\right) + \mathcal{O}(\zeta^4, v^4). \tag{41}$$

- And hence

$$p(\zeta,v) = \frac{1}{2}\left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial \zeta^2}\right)\mathcal{R}(\zeta,v) = \frac{1}{4}\left(1+\Delta\right) + \mathcal{O}(\zeta^2, v^2). \tag{42}$$

- And finally

$$p(\zeta) = \frac{1}{2}\left(1+\Delta\right)\zeta + \mathcal{O}\left(\zeta^3\right) \ , \tag{43}$$

- To be compared with

$$p_{\rm uncorr}(\zeta) \sim \left(\frac{2}{3} + \Delta\right) \zeta \ , \text{when } \zeta \to 0 \ . \tag{44}$$

$\zeta \to \infty$ Asymptotics: Critical

- we obtain

$$\mathcal{R}(\zeta, 0) = \frac{\mathcal{B}}{\zeta^2} + \mathcal{O}\left(\frac{1}{\zeta^4}\right), \tag{45}$$

where

$$\mathcal{B} = 6\, \mathcal{C}^{*2} \approx 35.3901, \qquad \text{with} \qquad \mathcal{C}^* = \frac{\sqrt{\pi}}{3} \frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})}. \tag{46}$$

- Hence, in the scaling limit $\zeta \to \infty$, $v \to \infty$ keeping ζ / v fixed

$$\mathcal{G}\left(0, \frac{\mathcal{R}(\zeta, v)}{\mathcal{R}(\zeta, 0)}\right) = \mathcal{C}^* \frac{v}{\zeta}\ . \tag{47}$$

- Inverting the above Eq. (47), we get

$$\frac{\mathcal{R}(\zeta, v)}{\mathcal{R}(\zeta, 0)} = \mathcal{F}\left(\mathcal{C}^* \frac{v}{\zeta}\right), \tag{48}$$

where $\mathcal{F}(z)$ is defined as the inverse function of $\mathcal{G}(0, z)$.

- $$\mathcal{R}(\zeta, v) = \frac{\mathcal{B}}{\zeta^2} \mathcal{F}\left(\mathcal{C}^* \frac{v}{\zeta}\right). \quad (49)$$

  Inserting this expression into the expression for $p(\zeta, v)$ yields

  $$p(\zeta, v) = -\frac{\mathcal{B}}{2}\Big[\frac{6}{\zeta^4}\mathcal{F}\left(\mathcal{C}^* \frac{v}{\zeta}\right) + 6\mathcal{C}^* \frac{v}{\zeta^5}\mathcal{F}'\left(\mathcal{C}^* \frac{v}{\zeta}\right) + \mathcal{C}^{*2}\left(\frac{v^2}{\zeta^6} - \frac{1}{\zeta^4}\right)\mathcal{F}''\left(\mathcal{C}^* \frac{v}{\zeta}\right)\Big]. \quad (50)$$

- The span distribution is then

  $$p(\zeta) = -\frac{1}{\zeta^3}\left(\frac{\mathcal{B}}{\mathcal{C}^*}\right)\int_0^{\mathcal{C}^*} dz\Big[6\mathcal{F}(z) + 6z\mathcal{F}'(z) + \left(z^2 - \mathcal{C}^{*2}\right)\mathcal{F}''(z)\Big]. \quad (51)$$

- Hence, we obtain

  $$p(\zeta) \sim \frac{\mathcal{A}}{\zeta^3} \qquad \text{for large } \zeta, \quad (52)$$

- We can integrate this exactly!

$$\mathcal{A} = 8\pi\sqrt{3} = 43.53118\ldots \, . \tag{53}$$

- Thus the leading asymptotic behavior for large ζ is

$$p(\zeta) \sim \frac{8\pi\sqrt{3}}{\zeta^3} \, . \tag{54}$$

- To be compared with

$$p_{\mathrm{uncorr}}(\zeta) \sim \frac{24}{\zeta^3}. \tag{55}$$

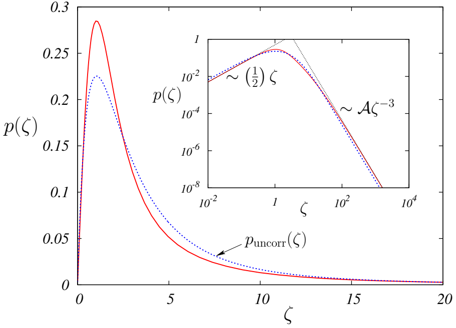


Figure: Theoretical stationary PDF of the dimensionless span $p(\zeta)$ (solid line) in the critical regime.

$\zeta \to \infty$ Asymptotics: Subcritical

- Here we get

$$p(\zeta) \sim \frac{A^2}{2}\,\zeta\,\exp\left(-\sqrt{\Delta}\,\zeta\right)\,, \quad \zeta \to \infty\,. \tag{56}$$

where $A = 12\,\Delta\,\exp\left[-2\sinh^{-1}\left(\sqrt{3\Delta/2}\right)\right]$.

- To be compared with

$$p_{\rm uncorr}(\zeta) \sim \frac{\Delta}{4} A^2\,\zeta\,\exp\left(-\sqrt{\Delta}\;\zeta\right)\,. \tag{57}$$

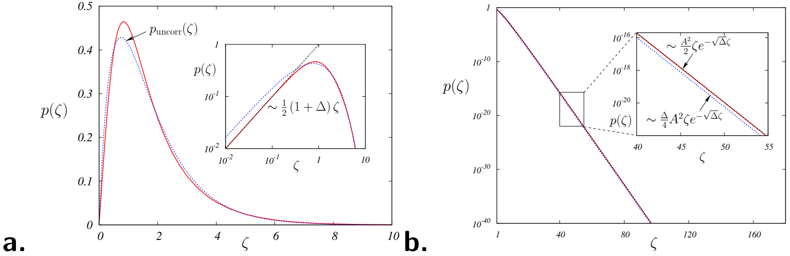


Figure: Theoretical stationary PDF of the dimensionless span $p(\zeta)$ in the subcritical regime.

Monte Carlo Simulations

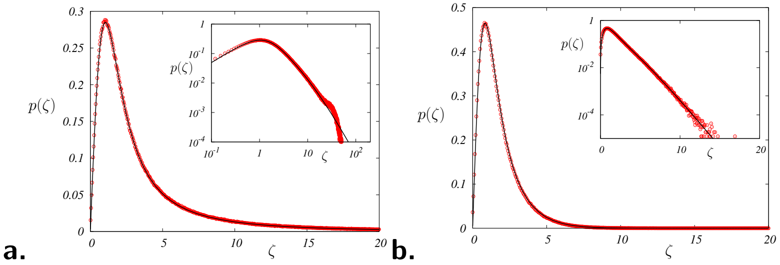


Figure: **a.** Probability distribution function of the dimensionless span $p(\zeta)$ extracted from Monte Carlo simulations (open circles) in the critical case ($\Delta = 0$). Here $t = 100$, $D = 1$, $a = b = 1$, and $dt = 0.0001$. The data is averaged over 5×10^7 realizations. **b.** Probability distribution function of the dimensionless span $p(\zeta)$ extracted from Monte Carlo simulations (open circles) in the subcritical regime. Here $t = 100$, $D = 1$, $a = 2$, $b = 1$ (i.e. $\Delta = 1$), and $dt = 0.0001$.

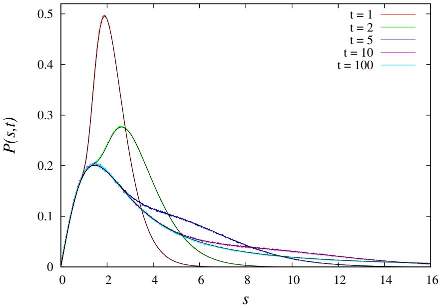


Figure: Finite time span PDF $P(s,t)$ obtained from Monte Carlo simulations at different times with $dt = 0.0001$, $D = 1$ and $b = 0.5$. The data is averaged over 5×10^7 realizations. The bold lines represent the PDFs obtained from our numerical integration of the two dimensional non-linear partial differential equation. We find a perfect agreement between the PDFs obtained by both techniques.

Conclusion

- We obtained exact **analytical results for the span distribution** of one dimensional BBM.
- This was possible by **looking at the stationary regime**.
- We found that **correlations between the maximum and minimum persist in the stationary regime**.
- It will be interesting to extend our analysis to **convex hulls in higher dimensional BBM**.

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