Universal Order and Gap Statistics of Critical Branching Brownian Motion: Supplementary Material

Kabir Ramola,[1,*] Satya N. Majumdar,[1,†] and Grégory Schehr[1,‡]

[1]CNRS, LPTMS, Univ. Paris-Sud, 91405 Orsay Cedex, France

(Dated: May 6, 2014)

In this supplementary material we provide additional details related to the critical branching Brownian motion discussed in our Letter.

I. NUMBER OF PARTICLES IN THE SYSTEM

$P(n,t)$ is defined as the probability that there are n particles in the system at time t, starting from a single particle at $x = 0$ at time $t = 0$. The normalization condition is

$$\sum_{n=0}^{\infty} P(n,t) = 1. \tag{1}$$

In order to derive the evolution equation for this quantity we use the backward Fokker-Planck (BFP) approach. We split the time interval $[0, t+\Delta t]$ into two subintervals: $[0,\Delta t]$ and $[\Delta t, t+\Delta t]$. We then look at the contribution of the terms generated at the first time step to the probability $P(n, t+\Delta t)$ at the final time step. In the first time interval $[0,\Delta t]$, the particle at $x = 0$ can:

A) branch into two walks with probability $b\Delta t$, resulting in two particles that give rise to r and $n-r$ particles at time $t+\Delta t$ respectively. The contribution from this branching term is then $b\Delta t \sum_{r=0}^{n} P(r,t)P(n-r,t)$.

B) die with a probability $d\Delta t$, leading to no particles at subsequent times, only contributing to $P(n = 0, t)$. The contribution from this term is thus $d\Delta t \delta_{n,0}$.

C) diffuse with probability $1 - (b + d)\Delta t$, with the single particle at time step Δt giving rise to n particles at time step $t+\Delta t$. The contribution from this term is thus $(1 - (b + d)\Delta t)\, P(n,t)$.

Adding the contribution from these terms, we have

$$\begin{aligned} P(n, t+\Delta t) = \delta_{n,0} d\Delta t + (1 - (b+d)\Delta t)P(n,t) \\ +b\Delta t \sum_{r=0}^{n} P(r,t)P(n-r,t). \end{aligned} \tag{2}$$

Expanding the above equation and taking the limit $\Delta t \to 0$, we obtain

$$\frac{dP(n,t)}{dt} = \delta_{n,0} d - (b+d)P(n,t) + b\sum_{r=0}^{n} P(r,t)P(n-r,t). \tag{3}$$

At the critical point $b = d$, this becomes

$$\frac{dP(n,t)}{dt} = b\left(\delta_{n,0} - 2P(n,t) + \sum_{r=0}^{n} P(r,t)P(n-r,t)\right). \tag{4}$$

In order to solve this equation we introduce the generating function

$$\mathcal{P}(\lambda,t) = \sum_{n=0}^{\infty} \lambda^n P(n,t). \tag{5}$$

This evolves in time according to

$$\frac{d\mathcal{P}(\lambda,t)}{dt} = b\left(\mathcal{P}(\lambda,t) - 1\right)^2, \tag{6}$$

with the initial condition $P(n,0) = \delta_{n,1}$ which translates to $\mathcal{P}(\lambda,0) = \lambda$. Solving the above equation we find

$$\mathcal{P}(\lambda,t) = 1 + \frac{1}{1/(\lambda-1) - bt}, \tag{7}$$

from which we extract the individual probabilities

$$\begin{aligned} P(0,t) &= \frac{bt}{1+bt}, \\ P(n\geq 1,t) &= \frac{(bt)^{n-1}}{(1+bt)^{n+1}}, \end{aligned} \tag{8}$$

which is Eq. (1) in the main text.

II. STATISTICS OF THE RIGHTMOST PARTICLE

A. Backward Fokker-Planck equation for $Q(n;x,t)$

$Q(n;x,t)$ is defined as the probability that starting with one particle at position $x = 0$ at time $t = 0$, there are $n \geq 1$ particles in the system at time t, with all of them lying to the left of x. To derive the evolution equation for this quantity, we use the BFP approach as in Section I. In the time interval $[0,\Delta t]$, the particle at

*kabir.ramola@u-psud.fr
†majumdar@lptms.u-psud.fr
‡gregory.schehr@lptms.u-psud.fr

$x = 0$ can:

A) split with probability $b\Delta t$, resulting in two particles that give rise to $r \geq 1$ and $n - r \geq 1$ particles at time $t + \Delta t$ respectively, with all the particles lying to the left of x. Either of these particles can also give rise to no particles at the final time with the other process giving rise to n particles. The contribution from the branching term is then $b\Delta t P(0,t)Q(n;x,t) + b\Delta t \sum_{r=1}^{n-1} Q(r;x,t)Q(n-r;x,t)$.

B) die with a probability $d\Delta t$. Since there are no particles at subsequent times this does not contribute to the probability $Q(n;x,t+\Delta t)$.

C) diffuse with probability $1-(b+d)\Delta t$, moving a distance $\Delta x = \eta(0)\Delta t$ in the first time step. This shifts the process by a distance Δx at the first time step. The contribution from this term is then $(1-(b+d)\Delta t)\,\langle Q(n;x-\eta(0)\Delta t,t)\rangle_{\eta(0)}$. Here, and in the following, $\langle \ldots \rangle_{\eta(0)}$ denotes an average over all possible values of the diffusive jump at the first time step.

Adding the contribution from these terms, we have

$$Q(n;x,t+\Delta t) = (1-(b+d)\Delta t)\,\langle Q(n;x-\eta(0)\Delta t,t)\rangle_{\eta(0)}$$
$$+2b\Delta t P(0,t)Q(n;x,t) + b\Delta t \sum_{r=1}^{n-1} Q(r;x,t)Q(n-r;x,t). \tag{9}$$

Using the properties of the Brownian noise

$$\langle \eta(0)\rangle = 0,$$
$$\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t'), \tag{10}$$

we can Taylor expand Eq. (9) up to second order in Δt, and arrive at

$$\frac{\partial Q(n;x,t)}{\partial t} = D\frac{\partial^2 Q(n;x,t)}{\partial x^2} - (b+d)Q(n;x,t)$$
$$+2bP(0,t)Q(n;x,t) + b\sum_{r=1}^{n-1} Q(r;x,t)Q(n-r;x,t). \tag{11}$$

At the critical point $b = d$, this equation reduces to

$$\frac{\partial Q(n;x,t)}{\partial t} = D\frac{\partial^2 Q(n;x,t)}{\partial x^2} - 2bQ(n;x,t)$$
$$+2bP(0,t)Q(n;x,t) + b\sum_{r=1}^{n-1} Q(r;x,t)Q(n-r;x,t)\ , \tag{12}$$

which is Eq. (2) in the main text. Using the expression in Eq. (8) for the value of $P(0,t) = bt/(1+bt)$ at the critical point, we arrive at

$$\frac{\partial Q(n;x,t)}{\partial t} = D\frac{\partial^2 Q(n;x,t)}{\partial x^2} - \frac{2b}{1+bt}Q(n;x,t)$$
$$+b\sum_{r=1}^{n-1} Q(r;x,t)Q(n-r;x,t). \tag{13}$$

B. Exact expression for $n = 1$

Using Eq. (13), we have the following evolution equation for $n = 1$

$$\frac{\partial Q(1;x,t)}{\partial t} = D\frac{\partial^2 Q(1;x,t)}{\partial x^2} - \frac{2b}{1+bt}Q(1;x,t), \tag{14}$$

with the initial condition

$$Q(1;x,0) = \Theta(x), \tag{15}$$

where $\Theta(x)$ is the Heaviside step function. We make the transformation

$$Q(n;x,t) = e^{-\int dt \frac{2b}{1+bt}} Q^\circ(n;x,t) = \frac{1}{(1+bt)^2} Q^\circ(n;x,t). \tag{16}$$

This transformation is used several times in the subsequent discussions (we use the convention that the superscript $^\circ$ denotes a multiplication by the factor $(1+bt)^2$). This removes the linear term in Eq. (14). We therefore have

$$\frac{\partial Q^\circ(1;x,t)}{\partial t} = D\frac{\partial^2 Q^\circ(1;x,t)}{\partial x^2}. \tag{17}$$

We recall here that the general diffusion equation with a time-dependent source term

$$\frac{\partial}{\partial t} f(x,t) = D\frac{\partial^2}{\partial x^2} f(x,t) + s(x,t), \tag{18}$$

can be solved as

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-x')^2}{4Dt}\right) f(x',0) \tag{19}$$
$$+ \int_0^t \frac{dt'}{\sqrt{4\pi D(t-t')}} \int_{-\infty}^{\infty} dx' \exp\left(-\frac{(x-x')^2}{4D(t-t')}\right) s(x',t').$$

We use this expression to solve Eq. (17) (which is a simple diffusion equation without a source). The initial condition is $Q^\circ(1;x,0) = \Theta(x)$. We then have

$$\begin{aligned} Q^\circ(1;x,t) &= \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-x')^2}{4Dt}\right) Q^\circ(1;x',0) \\ &= \int_0^{\infty} \frac{dx'}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-x')^2}{4Dt}\right) \\ &= \frac{1}{2}\mathrm{erfc}\left(\frac{-x}{\sqrt{4Dt}}\right). \end{aligned} \tag{20}$$

where $\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-t^2)dt$ is the complementary error function. The probability conditioned on the fact that there is one particle in the system is then

$$\begin{aligned} Q(x,t|1) = \frac{Q(1;x,t)}{P(1,t)} &= \frac{1}{(1+bt)^2}\frac{Q^\circ(1;x,t)}{P(1,t)} \\ &= \frac{1}{2}\mathrm{erfc}\left(\frac{-x}{\sqrt{4Dt}}\right). \end{aligned} \tag{21}$$

Above we have used the fact that $P(1,t) = 1/(1+bt)^2$ from Eq. (8).

C. Exact expression for $n = 2$

Using Eq. (13), $Q(2; x, t)$ evolves according to

$$\frac{\partial Q(2;x,t)}{\partial t} = D\frac{\partial^2 Q(2;x,t)}{\partial x^2} - \frac{2b}{1+bt}Q(2;x,t) + b\left(Q(1;x,t)\right)^2. \quad (22)$$

Using the transformation in Eq. (16) we have

$$\frac{\partial Q^\circ(2;x,t)}{\partial t} = D\frac{\partial^2 Q^\circ(2;x,t)}{\partial x^2} + \frac{b}{(1+bt)^2}(Q^\circ(1;x,t))^2. \quad (23)$$

Using Eq. (19) and the initial condition $Q(2; x, 0) = 0$ (since there is only one particle in the system at $t = 0$), we arrive at the following exact expression for the conditional probability

$$Q(x,t|2) = \frac{Q(2;x,t)}{P(2,t)} = \left(\frac{1+bt}{bt}\right)\int_0^t \frac{bdt'}{(1+bt')^2}\int_{-\infty}^{\infty} dx' \frac{1}{\sqrt{4\pi D(t-t')}} \times \exp\left(-\frac{(x-x')^2}{4D(t-t')}\right)\frac{1}{4}\mathrm{erfc}^2\left(-\frac{x'}{\sqrt{4Dt'}}\right). \quad (24)$$

We have used $P(2, t) = bt/(1 + bt)^3$ from Eq. (8) above. In order to analyze the large time behavior of this expression, we make the following change of variables

$$z = \frac{x}{\sqrt{4Dt}} \quad \text{with} \quad \begin{cases} x' = x\xi, \\ t' = t\tau, \end{cases}$$

We then have the following expression in terms of the scaled variables

$$Q(x,t|2) = \left(\frac{1+bt}{bt}\right)\left(\frac{z}{t}\right) \times \Big[\int_0^1 \frac{bd\tau}{(\frac{1}{t}+b\tau)^2}\int_{-\infty}^{\infty} d\xi \exp\left(-z^2\frac{(1-\xi)^2}{(1-\tau)}\right) \frac{1}{\sqrt{\pi(1-\tau)}}\frac{1}{4}\mathrm{erfc}^2\left(-z\frac{\xi}{\sqrt{\tau}}\right)\Big]. \quad (25)$$

The major contribution to this integral arises from the the region $\tau \to 0$. Here, the $\frac{1}{t}$ in the denominator acts as a regularization parameter. In order to estimate this integral we break up the τ integral into a divergent part $[0, \epsilon]$ which diverges as $\left(\frac{t}{z}\right)$ at large times, and a regular part $[\epsilon, \infty]$ that converges for all $\epsilon > 0$. The contribution from the divergent term can be estimated as follows. For small ϵ we can expand the complementary error function as

$$\frac{1}{4}\mathrm{erfc}^2\left(-z\frac{\xi}{\sqrt{\tau}}\right) = \Theta(\xi) + \mathcal{O}\left(\exp\left(-\xi\frac{z}{\sqrt{\epsilon}}\right)\right). \quad (26)$$

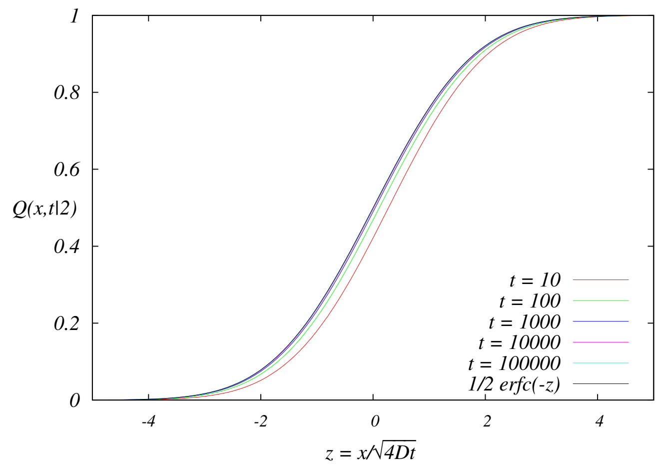

FIG. 1: Plot of $Q(x,t|2)$ computed from the integral given in Eq. (25) for different values of t, showing the convergence to the long time behavior of Eq. (28).

We can then easily perform the integral over ξ. We then have, for large t

$$Q(x,t|2) \sim \left(\frac{1}{t}\right)\left[\int_0^\epsilon \frac{bd\tau}{(\frac{1}{t}+b\tau)^2}\frac{1}{2}\mathrm{erfc}\left(-\frac{z}{\sqrt{1-\tau}}\right)\right] \quad (27)$$

Taking the limit $\epsilon \to 0$ gives us the desired result at large times t

$$Q(x,t|2) = \frac{1}{2}\mathrm{erfc}\left(-\frac{x}{\sqrt{4Dt}}\right). \quad (28)$$

We can also numerically integrate the expression for $Q(x,t|2)$ to obtain the behavior at large times. In Fig. 1 we plot the integral given in Eq. (24) for different values of t, showing the convergence to the long time behavior of Eq. (28).

D. Asymptotic behavior for $n \geq 1$

For general $n \geq 1$ we have the following evolution equation (Eq. 13, which we repeat below)

$$\frac{\partial Q(n;x,t)}{\partial t} = D\frac{\partial^2 Q(n;x,t)}{\partial x^2} - \frac{2b}{1+bt}Q(n;x,t) + b\sum_{r=1}^{n-1} Q(r;x,t)Q(n-r;x,t). \quad (29)$$

Using the transformation in Eq. (16) we have

$$\frac{\partial Q^\circ(n;x,t)}{\partial t} = D\frac{\partial^2 Q^\circ(n;x,t)}{\partial x^2} + \frac{b}{(1+bt)^2}\sum_{r=1}^{n-1} Q^\circ(r;x,t)Q^\circ(n-r;x,t). \quad (30)$$

Using Eqs. (8) and (16), the probability conditioned on the number of particles n can be expressed as

$$Q(x,t|n) = \frac{Q(n;x,t)}{P(n,t)} = \left(\frac{1+bt}{bt}\right)^{n-1} Q^{\circ}(n;x,t). \quad (31)$$

The equation for the evolution of $Q(x,t|n)$ is then

$$\frac{\partial}{\partial t}\left(\left(\frac{bt}{1+bt}\right)^{n-1} Q(x,t|n)\right) = \left(\frac{bt}{1+bt}\right)^{n-1} D\frac{\partial^2 Q(x,t|n)}{\partial x^2} + \left(\frac{bt}{1+bt}\right)^{n-2} \frac{b}{(1+bt)^2}\sum_{r=1}^{n-1} Q(x,t|r)Q(x,t|n-r).$$

Simplifying the above equation leads to

$$\frac{\partial Q(x,t|n)}{\partial t} + \frac{n-1}{t(1+bt)}Q(x,t|n) = D\frac{\partial^2 Q(x,t|n)}{\partial x^2} + \frac{1}{t(1+bt)}\sum_{r=1}^{n-1} Q(x,t|r)Q(x,t|n-r), \quad (32)$$

which is Eq. (3) described in the main text. $Q(x,t|n)$ satisfies the boundary conditions

$$\begin{aligned} Q(x,t|n) &= 1 \quad \text{for} \quad x\to\infty, \\ &= 0 \quad \text{for} \quad x\to-\infty, \end{aligned} \quad (33)$$

and is bounded as $0 < Q(x,t|n) < 1$ in the domain $(-\infty,\infty)$. Thus, in the large time limit $Q(x,t|n)$ obeys the simple diffusion equation

$$\frac{\partial Q(x,t|n)}{\partial t} = D\frac{\partial^2 Q(x,t|n)}{\partial x^2}, \quad (34)$$

which has the solution

$$Q(x,t|n) = \frac{1}{2}\mathrm{erfc}\left(\frac{-x}{\sqrt{4Dt}}\right). \quad (35)$$

Note: Although the initial condition $Q(n;x,0) = 0$ for $n > 1$, the initial condition for the conditional probability is $Q(x,0|n) = \Theta(x)$.

III. GAP STATISTICS

A. One Particle Case

1. Backward Fokker-Planck equation for $P(1;x,t)$

$P(1;x,t)$ is defined as the joint probability distribution function (PDF) that there is one particle in the system at time t, which is at position x. To derive the evolution equation for this quantity we use the BFP approach. In the first time interval $[0,\Delta t]$, the particle at $x=0$ can:

A) split into two particles with probability $b\Delta t$ with one branch giving rise to the single particle at the final time and the other giving rise to no particles. The contribution from this term is then $2b\Delta t P(0,t)P(1;x,t)$.

B) die with a probability $d\Delta t$, leading to no particles at subsequent times and thus not contributing to the probability $P(1;x,t)$.

C) diffuse with probability $1-(b+d)\Delta t$ moving a distance $\Delta x = \eta(0)\Delta t$, with the single particle at time step Δt giving rise to one particle at time step $t+\Delta t$ at position x. The contribution from this term is thus $(1-(b+d)\Delta t)\, P(1;x-\eta(0)\Delta t,t)\rangle_{\eta(0)}$.

Adding the contribution from the different terms we arrive at

$$\begin{aligned} &P(1;x,t+\Delta t) = 2b\Delta t P(0,t)P(1;x,t) + \\ &(1-(b+d)\Delta t)\,\langle P(1;x-\eta(0)\Delta t,t)\rangle_{\eta(0)}. \end{aligned} \quad (36)$$

Expanding the above equation up to second order in Δt, using the properties of the noise in Eq. (10) and taking the limit $\Delta t\to 0$, we arrive at the following evolution equation for the PDF (for $b=d$)

$$\frac{\partial}{\partial t}P(1;x,t) = D\left(\frac{\partial}{\partial x}\right)^2 P(1;x,t) - \left(\frac{2b}{1+bt}\right)P(1;x,t). \quad (37)$$

Above we have used the expression $P(0,t) = bt/(1+bt)$ from Eq. (8).

2. Exact Solution

Following Eq. (16), we make the transformation

$$P(1;x,t) = \frac{1}{(1+bt)^2}P^{\circ}(1;x,t), \quad (38)$$

leading to

$$\frac{\partial}{\partial t}P^{\circ}(1;x,t) = D\left(\frac{\partial}{\partial x}\right)^2 P^{\circ}(1;x,t). \quad (39)$$

Using Eq. (19) and the fact that $P^{\circ}(1;x,0) = \delta(x)$ we have

$$P(1;x,t) = \frac{1}{(1+bt)^2}\frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right). \quad (40)$$

This is Eq. (4) in the main text. This can also be arrived at by using the relation

$$P(1;x,t) = \frac{\partial}{\partial x} Q(1;x,t). \tag{41}$$

Inserting the expression from Eq. (20) for $Q(1;x,t)$ in the above equation, we recover the solution in Eq. (40). The PDF of the position of the particle conditioned on the fact that there is one particle in the system is then

$$P(x,t|1) = \frac{P(1;x,t)}{P(1,t)} = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right), \tag{42}$$

where we have used $P(1,t) = 1/(1+bt)^2$ from Eq. (8).

B. Two Particle Case

1. Backward Fokker-Planck equation for $P(2;x_1,x_2,t)$

$P(2;x_1,x_2,t)$ is defined as the joint PDF that there are two particles in the system at time t, with the first at position x_1 and the second at $x_2 < x_1$. To derive the evolution equation for this quantity we once again employ the BFP approach. In the first time interval $[0,\Delta t]$, the particle at $x=0$ can:

A) split into two particles with probability $b\Delta t$. There are two cases to consider:

- one branch gives rise to two particles at positions x_1 and x_2 at the final time and the other gives rise to no particles. The contribution from this term is then $2b\Delta t P(0,t) P(2;x_1,x_2,t)$.
- one branch gives rise to one particle at the final time at position x_1 and the other gives rise to a single particle at position x_2. The contribution from this term is then $2b\Delta t P(1;x_1,t)P(1;x_2,t)$.

B) die with a probability $d\Delta t$, leading to no particles at subsequent times and thus not contributing to the probability $P(2;x_1,x_2,t)$.

C) diffuse with probability $1-(b+d)\Delta t$ moving a distance $\Delta x = \eta(0)\Delta t$, with the single particle at time step Δt giving rise to two particles at positions x_1 and x_2 at time step $t+\Delta t$. The contribution from this term is thus $(1-(b+d)\Delta t)\, P(2;x_1-\eta(0)\Delta t, x_2-\eta(0)\Delta t, t)\rangle_{\eta(0)}$.

Adding the contribution from the different terms we arrive at

$$\begin{aligned}&P(2;x_1,x_2,t+\Delta t) = 2b\Delta t P(0,t)P(2;x_1,x_2,t)\\&+2b\Delta t P(1;x_1,t)P(1;x_2,t)\\&+(1-(b+d)\Delta t)\,\langle P(2;x_1-\eta(0)\Delta t, x_2-\eta(0)\Delta t,t)\rangle_{\eta(0)}.\end{aligned} \tag{43}$$

Expanding the above equation up to second order in Δt, using the properties of the noise in Eq. (10) and taking the limit $\Delta t \to 0$, we arrive at the following evolution equation for the PDF (for $b=d$)

$$\begin{aligned}&\frac{\partial}{\partial t}P(2;x_1,x_2,t) = D\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)^2 P(2;x_1,x_2,t)\\&-\left(\frac{2b}{1+bt}\right)P(2;x_1,x_2,t) + 2bP(1;x_2,t)P(1;x_2,t).\end{aligned} \tag{44}$$

Above, we have used $P(0,t) = bt/(1+bt)$ from Eq. (8). This is Eq. (5) described in the main text.

2. Exact Solution

Following Eq. (16), we make the transformation

$$P(2;x_1,x_2,t) = \frac{1}{(1+bt)^2} P^\circ(2;x_1,x_2,t). \tag{45}$$

$P^\circ(2;x_1,x_2,t)$ then satisfies the evolution equation

$$\begin{aligned}&\frac{\partial}{\partial t}P^\circ(2;x_1,x_2,t) = D\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)^2 P^\circ(2;x_1,x_2,t).\\&+2b(1+bt)^2 P(1;x_1,t)P(1;x_2,t).\end{aligned} \tag{46}$$

We next make the change of variables

$$\begin{aligned} s &= \frac{x_1+x_2}{2},\\ g_1 &= x_1 - x_2 > 0,\\ P^\circ(2;x_1,x_2,t) &\to \widetilde{P}^\circ(2;s,g_1,t).\end{aligned} \tag{47}$$

s and g_1 have been described in the main text as the centre of mass and gap variables respectively. The Jacobian of this transformation is 1. Using the expression for $P(1;x,t)$ from Eq. (40), we have

$$\begin{aligned}&\frac{\partial}{\partial t}\widetilde{P}^\circ(2;s,g_1,t) = D\left(\frac{\partial}{\partial s}\right)^2 \widetilde{P}^\circ(2;s,g_1,t)\\&+\frac{2b}{(1+bt)^2}\frac{1}{4\pi Dt}\exp\left(-\frac{2s^2+\frac{1}{2}g_1^2}{4Dt}\right).\end{aligned} \tag{48}$$

This is a diffusion equation with a time-dependent source term. Using Eq. (19) and the initial condition $\widetilde{P}^\circ(2;s,g_1,t)=0$, we arrive at the following exact solution

$$\begin{aligned}&\widetilde{P}^\circ(2;s,g_1,t) =\\&\int_0^t dt' \int_{-\infty}^{\infty} ds' \frac{1}{\sqrt{4\pi D(t-t')}}\exp\left(-\frac{(s'-s)^2}{4D(t-t')}\right)\times\\&\frac{2b}{(1+bt')^2}\frac{1}{4\pi Dt'}\exp\left(-\frac{2s'^2+\frac{1}{2}g_1^2}{4Dt'}\right).\end{aligned} \tag{49}$$

The conditional PDF that given there are two particles in the system at time t, their centre of mass and the gap between them are s and g_1 respectively is

$$\widetilde{P}(s, g_1, t|2) = \frac{\widetilde{P}(2; s, g_1, t)}{P(2, t)}. \quad (50)$$

Using Eq. (45) and the fact that $P(2, t) = bt/(1 + bt)^3$ from Eq. (8), we have

$$\widetilde{P}(s, g_1, t|2) = \left(\frac{1 + bt}{bt}\right) \widetilde{P}^\circ(2; s, g_1, t). \quad (51)$$

Integrating Eq. (49) with respect to s' and using Eq. (51) we arrive at

$$\widetilde{P}(s, g_1, t|2) = \left(\frac{1 + bt}{2\pi Dt}\right) \int_0^t \frac{dt'}{(1 + bt')^2} \frac{e^{-\frac{g_1^2}{8Dt'} - \frac{s^2}{2D(2t-t')}}}{\sqrt{t'(2t - t')}}. \quad (52)$$

This is Eq. (6) in the main text. Integrating over the centre of mass variable s in Eq. (52), we arrive at the marginal PDF of the gap

$$\widetilde{P}(g_1, t|2) = \left(\frac{1 + bt}{bt}\right) \int_0^t \frac{bdt'}{(1 + bt')^2} \frac{\exp(-\frac{g_1^2}{8Dt'})}{\sqrt{2\pi Dt'}} , \quad (53)$$

which is Eq. (7) in the main text. Using this expression we can derive the stationary distribution $p(g_1|2)$ by taking the limit $t \to \infty$. We have

$$p(g_1|2) = \int_0^\infty \frac{bdt'}{(1 + bt')^2} \frac{\exp(-\frac{g_1^2}{8Dt'})}{\sqrt{2\pi Dt'}}. \quad (54)$$

It is possible to perform this integral exactly. We have $p(g_1|2) = (4\sqrt{D/b})^{-1} f[g_1/(4\sqrt{D/b})]$ with

$$f(x) = -4x + \sqrt{2\pi}\, e^{2x^2} (1 + 4x^2) \operatorname{erfc}(\sqrt{2}\, x) . \quad (55)$$

This is Eq. (8) in the main text.

C. General $n > 2$

1. Backward Fokker-Planck equation for $P(n; x_1, x_2, t)$

$P(n; x_1, x_2, t)$ is defined as the joint PDF that there are n particles in the system at time t, with the first at position x_1 and the second at position x_2. To derive the evolution equation for this quantity we once again employ the BFP approach. In the first time interval $[0, \Delta t]$, the particle at $x = 0$ can:

A) split into two particles with probability $b\Delta t$. There are three different cases to consider:

- One branch gives rise to 0 particles while the other gives rise to n particles. This term has the contribution $2b\Delta t P(0, t)P(n; x_1, x_2, t)$
- One branch gives rise to 1 particle while the other gives rise to $n - 1$ particles. The first two particles from the $n-1$ particle branch and the particle from the 1 particle branch are ordered as $x_1 > x_2 > x_3$ at the final time step, with any of them belonging to either branch. This term therefore has a contribution $2b\Delta t \int_{-\infty}^{x_2} dx_3 \sum_{\sigma\in S_3} P(1; x_{\sigma_1}, t)P(n - 1; x_{\sigma_2}, x_{\sigma_3}, t)$. Here $\sum_{\sigma\in S_N}$ denotes a sum over the permutations σ of N elements and $\sigma_i \equiv \sigma(i)$ and with the convention that $P(r; x_i, x_j, t) = 0$ for $i > j$. We note that although x_1, and x_2 are the positions of the first two particles, x_3 is not necessarily the position of the third particle at the edge of the system.
- One branch gives rise to $r \geq 2$ particles while the other gives rise to $n - r \geq 2$. This term therefore has a contribution $b\Delta t \sum_{r=2}^{n-2} \int_{-\infty}^{x_2} dx_3 \int_{-\infty}^{x_3} dx_4 \sum_{\sigma\in S_4} \times P(r; x_{\sigma_1}, x_{\sigma_3}, t)P(n - r; x_{\sigma_3}, x_{\sigma_3}, t)$.

B) die with a probability $d\Delta t$, leading to no particles at subsequent times and thus not contributing to the probability $P(n; x_1, x_2, t)$.

C) diffuse with probability $1 - (b + d)\Delta t$, with the single particle at time step Δt giving rise to n particles at time step $t + \Delta t$. The contribution from this term is thus $(1 - (b + d)\Delta t)\, P(n; x_1 - \eta(0)\Delta t, x_2 - \eta(0)\Delta t, t)\rangle_{\eta(0)}$.

Adding the contribution from all these terms, we have

$$\begin{aligned} &P(n; x_1, x_2, t + \Delta t) = \\ &(1 - (b + d)\Delta t) \langle P(n; x_1 - \eta(0)\Delta t, x_2 - \eta(0)\Delta t, t)\rangle_{\eta(0)} \\ &\qquad + 2b\Delta t P(0, t)P(n; x_1, x_2, t) \\ &\qquad + b\Delta t \mathcal{S}(n; x_1, x_2, t), \end{aligned} \quad (56)$$

where $\mathcal{S}(n; x_1, x_2, t) \equiv \mathcal{S}$ is

$$\begin{aligned} \mathcal{S} = \int_{-\infty}^{x_2} dx_3 \Big[2 \sum_{\sigma\in S_3} P(1; x_{\sigma_1}, t)P(n - 1; x_{\sigma_2}, x_{\sigma_3}, t) \quad (57) \\ + \sum_{r=2}^{n-2} \int_{-\infty}^{x_3} dx_4 \sum_{\sigma\in S_4} P(r; x_{\sigma_1}, x_{\sigma_2}, t)P(n - r; x_{\sigma_3}, x_{\sigma_4}, t) \Big]. \end{aligned}$$

We note that while x_1 and x_2 stand respectively for the positions of the first and second particle, x_3 and x_4 are not necessarily the positions of the third and fourth ones. Expanding Eq. (56) up to second order in Δt and using the properties of the noise in Eq. (10) we have (for $b = d$)

$$\begin{aligned} \frac{\partial P(n; x_1, x_2, t)}{\partial t} &= D \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2 P(n; x_1, x_2, t) \\ -&\frac{2b}{1 + bt} P(n; x_1, x_2, t) + b\mathcal{S}(n; x_1, x_2, t). \end{aligned} \quad (58)$$

This yields Eq. (10) in the main text, together with the explicit expression of the source term $\mathcal{S}$ (57).

2. *Three Particle Case*

Using Eq. (58) for $n = 3$, we have the following evolution equation

$$\frac{\partial}{\partial t}P(3;x_1,x_2,t) = D\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)^2 P(3;x_1,x_2,t)$$
$$-\left(\frac{2b}{1+bt}\right)P(3;x_1,x_2,t)$$
$$+2b\int_{-\infty}^{x_2} dx_3\Big[P(1;x_1,t)P(2;x_2,x_3,t)$$
$$+P(1;x_2,t)P(2;x_1,x_3,t)+P(1;x_3,t)P(2;x_1,x_2,t)\Big]$$

Using the procedure outlined in the previous section, we obtain the exact expression for the marginal PDF of the gap between the first two particles conditioned on three particles in the system stated in Eq. (60). It is interesting to note that due to symmetry, the PDF of the gap between the first and the second particles is the same as the PDF of the gap between the second and the third particle. In Fig. 4 we plot $\widetilde{P}(g_1,t|3)$ computed using the above expression along with data obtained from directly simulating the process. We also plot the PDF for large times showing a convergence to the steady state behavior with the the power law tail $8(\frac{D}{b})\frac{1}{g_1^3}$. In Fig. 2 we plot the contribution of the different terms in the source function to the marginal PDF $\widetilde{P}(g_1,t|3)$. We see that only the term $\int_{-\infty}^{x_2} P(1;x_1,t)P(2;x_2,x_3,t)dx_3$ has a significant contribution in the large gap limit, $x_1 - x_2 \gg 1$.

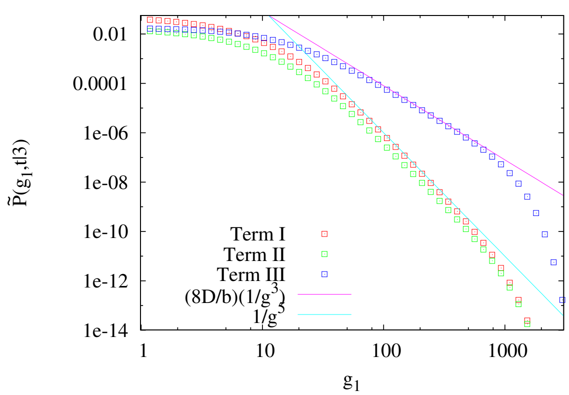

FIG. 2: Contributions to the marginal PDF $\widetilde{P}(g_1,t|3)$ from the three terms in the source function $P(1;x_3,t)P(2;x_1,x_2,t)$, $P(1;x_2,t)P(2;x_1,x_3,t)$ and $P(1;x_1,t)P(2;x_2,x_3,t)$ [integrated over $x_3 \in (-\infty, x_2]$ and over the center of mass $s = (x_1+x_2)/2 \in (-\infty,+\infty)$] respectively. We see that in the large gap limit, only the third term has a significant contribution to the marginal PDF. The other terms are suppressed by a factor $1/g_1^2$. ($D = 1, b = 1/2, t = 10^5$).

3. *Four Particle Case*

Using Eq. (58) for $n = 4$, we have the following evolution equation

$$\frac{\partial}{\partial t}P(4;x_1,x_2,t) = D\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)^2 P(4;x_1,x_2,t)$$
$$-\left(\frac{2b}{1+bt}\right)P(4;x_1,x_2,t)$$
$$+2b\int_{-\infty}^{x_2} dx_3\Big[P(1;x_1,t)P(3;x_2,x_3,t)$$
$$+P(1;x_2,t)P(3;x_1,x_3,t)$$
$$+P(1;x_3,t)P(3;x_1,x_2,t)\Big]$$
$$+2b\int_{-\infty}^{x_2} dx_3\int_{-\infty}^{x_3} dx_4\Big[P(2;x_1,x_2,t)P(2;x_3,x_4,t)$$
$$+P(2;x_1,x_3,t)P(2;x_2,x_4,t)$$
$$+P(2;x_1,x_4,t)P(2;x_2,x_3,t)\Big]. \quad (59)$$

This can be solved in a similar manner to the three particle case. In this case, the exact expression is very large and we do not present it here. In Fig 5 we plot the PDF of the first gap conditioned on four particles in the system obtained from Monte Carlo simulations along with our theoretical prediction at different times. We also plot the large time behavior of this function, showing the convergence to the asymptotic behavior $8(\frac{D}{b})\frac{1}{g_1^3}$ at large times.

$$\widetilde{P}(g_1,t|3) = 2\left(\frac{1+bt}{bt}\right)^2\int_0^t \frac{bdt_1}{(1+bt_1)^2}\int_0^{t_1}\frac{bdt_2}{(1+bt_2)^2}\times \quad (60)$$
$$\left(\frac{e^{-\frac{g_1^2}{8Dt_2}}\operatorname{erfc}\left[\frac{g_1}{2\sqrt{D(8t_1-2t_2)}}\right]}{\sqrt{8\pi Dt_2}}+\frac{e^{-\frac{g_1^2}{8Dt_1}}\operatorname{erfc}\left[-\frac{g_1(-2t_1+t_2)}{2\sqrt{2}\sqrt{Dt_1(4t_1-t_2)t_2}}\right]}{\sqrt{8\pi Dt_1}}+\frac{e^{-\frac{g_1^2}{8Dt_1}}\operatorname{erfc}\left[\frac{1}{2}g_1\sqrt{\frac{t_2}{8Dt_1^2-2Dt_1t_2}}\right]}{\sqrt{8\pi Dt_1}}\right).$$

IV. ASYMPTOTIC BEHAVIOUR

The computations of the distribution of the first gap conditioned on three and four particles in the system are quite instructive as they allow us to analyze Eq. (58) in the large t and large gap g_1 limit for generic n as follows. The solution of (58) is a linear combination of solutions arising from individual terms present in the source function $\mathcal{S}$. From this one can show that the PDF of the first gap conditioned on n particles converges to a stationary distribution $\widetilde{P}(g_1, t \to \infty|n) = p(g_1|n)$. While the full PDF $p(g_1|n)$ depends on n (see also Fig. 3 in the main text), its tail is universal. This follows from the fact that the leading contribution to $\mathcal{S}$ in (58) when the gap $g_1 = x_1 - x_2 \gg 1$ is large arises from the term in the first line $2bP(1; x_1, t) \int_{-\infty}^{x_2} dx_3 P(n-1; x_2, x_3, t)$ which tends to $2bP(1; x_1, t)P(1; x_2, t)$ at large t (since the rightmost particle conditioned on $n-1$ particles in the system behaves as a free diffusive particle at large times). This is precisely the source term for the two-particle case analyzed in Eq. (44). One can show that all other terms in $\mathcal{S}$ involve a larger gap between particles generated by the same offspring walk and are thus suppressed by a factor $\int_{g_1}^{\infty} p(g'|k)dg' \sim 1/{g_1}^2$, $k < n$. This is illustrated in Fig. 2 for the three particle case. Therefore, when $g_1 \to \infty$ the tail of the PDF of the first gap for general n converges to that of the two-particle case, $p(g_1|n) \sim \left(\frac{8D}{b}\right) g_1^{-3}$, for all n.

A similar analysis yields the asymptotic behavior of the k-th gap $g_k(t) = x_k(t) - x_{k+1}(t)$. In this case, we study $P(n; x_k, x_{k+1}, t)$, the joint PDF that there are n particles at time t with the k-th particle at position x_k and the $(k+1)$-th particle at position x_{k+1}. This PDF once again satisfies a diffusion equation with a source term similar to (58), from which we can show that the PDF of the kth gap reaches a stationary distribution $\widetilde{P}(g_k, t \to \infty|n) = p(g_k|n)$. In the large gap limit, the dominant term in the source function is the one in which the first k particles belong to one of the offsprings generated at the first time step, and the subsequent $n-k$ particles belong to the other. This term tends to $2bP(1; x_k, t)P(1; x_{k+1}, t)$ at large t, as it involves the leftmost particle of the first process being at x_k and the rightmost particle of the other process being at x_{k+1}. As noticed before for g_1, all other terms involve a large gap between particles generated by the same offspring process and are hence suppressed. This in turn leads to the large gap stationary behavior $p(g_k|n) \sim \left(\frac{8D}{b}\right) g_k^{-3}$ for all k and n. Therefore *the tail of the PDFs of the gaps are universal and are independent of n and k.*

V. SIMULATION RESULTS

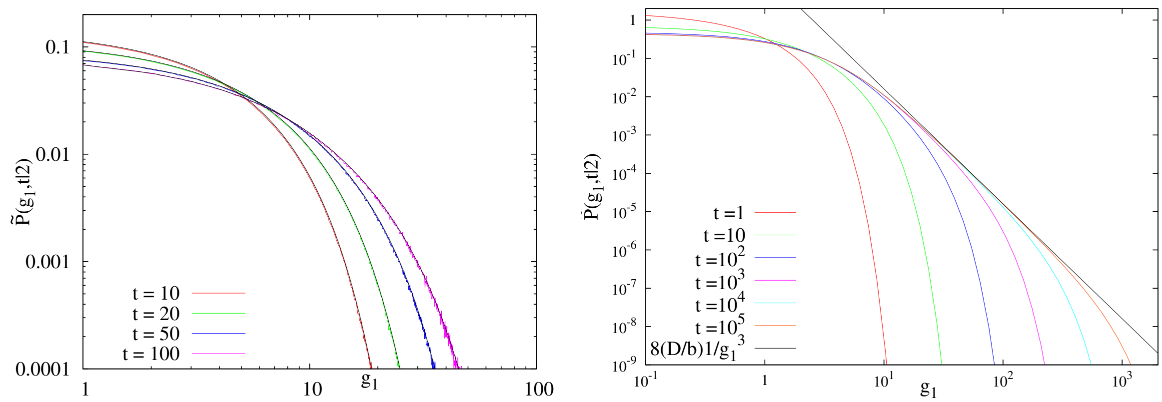

FIG. 3: (Left) The marginal PDF of the gap $\widetilde{P}(g_1, t|2)$ at different times conditioned on two particles obtained from Monte Carlo simulations (with 10^7 realizations). The thick lines correspond to the exact theoretical PDF (Eq. (53)). (Right) Theoretical PDF $\widetilde{P}(g_1, t|2)$ at large times showing the convergence to the stationary behavior.

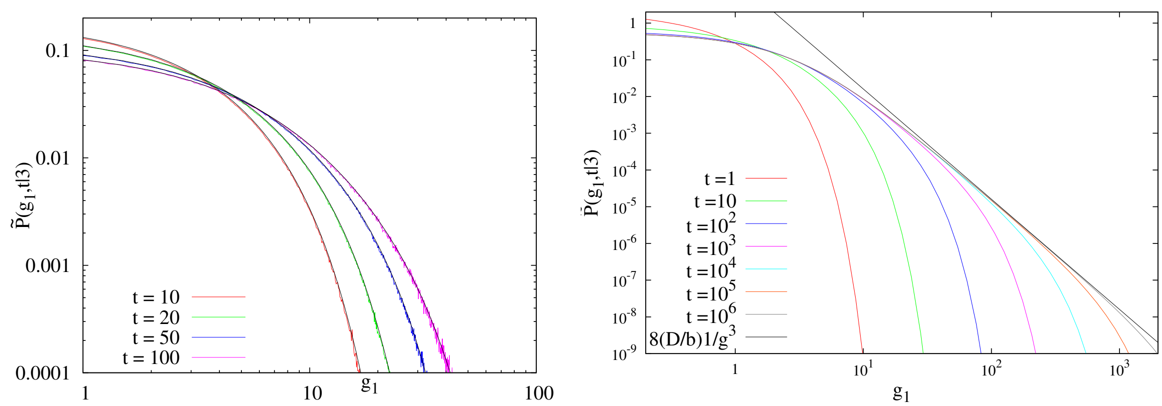

FIG. 4: (Left) The marginal PDF of the first gap $\widetilde{P}(g_1, t|3)$ at different times conditioned on three particles obtained from Monte Carlo simulations (with 10^7 realizations). The thick lines correspond to the exact theoretical PDF (Eq. (60)). (Right) The theoretical PDF $\widetilde{P}(g_1, t|3)$ at large times showing the convergence to the stationary behavior.

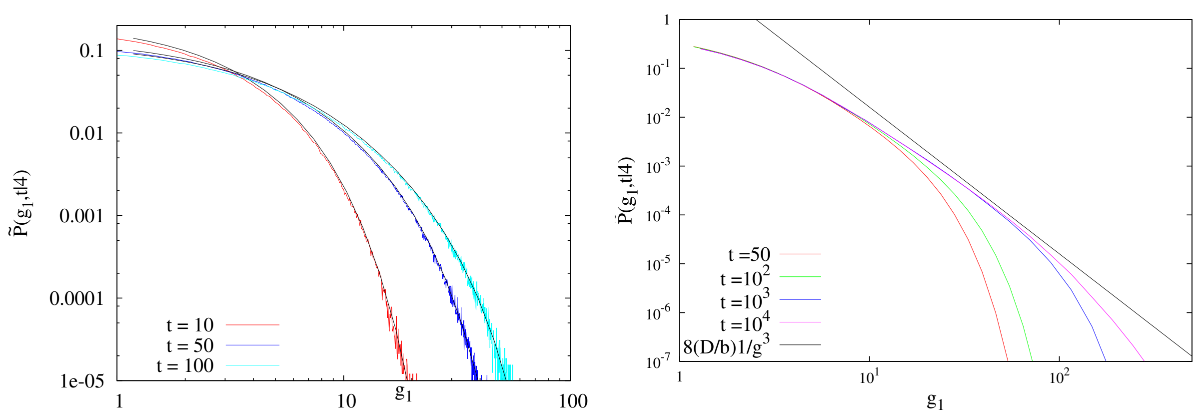

FIG. 5: (Left) The marginal PDF of the first gap $\widetilde{P}(g_1, t|4)$ at different times conditioned on four particles obtained from Monte Carlo simulations (with 10^7 realizations). The thick lines correspond to the exact theoretical PDF. (Right) The theoretical PDF $\widetilde{P}(g_1, t|4)$ at large times showing the convergence to the stationary behavior. (Numerically evaluating $\widetilde{P}(g_1, t|4)$ for very large times produces numerical error and we have displayed only up to $t = 10^4$ here.)