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Branching Brownian motion conditioned on particle numbers

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ABSTRACT

We study analytically the order and gap statistics of particles at time t for the one dimensional branching Brownian motion, conditioned to have a fixed number of particles at t. The dynamics of the process proceeds in continuous time where at each time step, every particle in the system either diffuses (with diffusion constant D), dies (with rate d) or splits into two independent particles (with rate b). We derive exact results for the probability distribution function of $g_k(t) = x_k(t) - x_{k+1}(t)$, the distance between successive particles, conditioned on the event that there are exactly n particles in the system at a given time t. We show that at large times these conditional distributions become stationary $P(g_k, t \to \infty | n) = p(g_k | n)$. We show that they are characterized by an exponential tail $p(g_k|n) \sim \exp\left[-\sqrt{\frac{|b-d|}{2D}} g_k\right]$ for large gaps in the subcritical ($b < d$) and supercritical ($b > d$) phases, and a power law tail $p(g_k) \sim 8\left(\frac{D}{b}\right) g_k^{-3}$ at the critical point ($b = d$), independently of n and k. Some of these results for the critical case were announced in a recent letter (Ramola et al., 2014).

1. Introduction

Branching processes are prototypical models of systems where new particles are generated at every time step – these include models of evolution, epidemic spreads and nuclear reactions amongst others [1–5]. An important model in this class is the Branching Brownian motion (BBM). We focus in this paper on the simple one-dimensional BBM, where the process starts with a single particle at the origin $x = 0$ at time $t = 0$. The dynamics proceeds in continuous time according to the following rules. In a small time interval Δt, each particle performs one of the three following microscopic moves: (i) it splits into two independent particles with probability $b\Delta t$, (ii) it dies with probability $d\Delta t$ and (iii) with the remaining probability $1 - (b + d)\Delta t$ it performs a Brownian motion moving by a stochastic distance $\Delta x(t) = \eta(t)\Delta t$. Here $\eta(t)$ is a Gaussian white noise with zero mean and delta-correlations with

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t_1)\eta(t_2) \rangle = 2D\delta(t_1 - t_2) \qquad (1)$$

where D is the diffusion constant. The delta function in the correlator (1) can be interpreted in the following sense: when $t_1 \neq t_2$, the noise is uncorrelated. In contrast, when $t_1 = t_2 = t$, the variance $\langle \eta^2(t) \rangle = 2D/\Delta t$. A realization of the dynamics of such a process is shown in Fig. 1. Depending on the parameters b and d, the average number of particles at time t in the system exhibits different asymptotic behaviors. For $b < d$, the *subcritical* phase, the process dies and on an average there are no particles at late times. For $b > d$, the *supercritical* phase, the process is explosive and the average number of particles grows exponentially with time t. In the borderline $b = d$ case, the system is critical, where on an average there is exactly one particle in the system at all times. This critical case is relevant to several physical and biological systems with stable population distributions [4].

BBM is a paradigmatic model of branching processes with wide applications and has been studied extensively in both mathematics and physics literature [1,4,6–8]. In

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one dimension, the positions of the particles at a particular time t represent a set of random variables that are naturally ordered according to their positions on the line with $x_1(t) > x_2(t) > x_3(t) \cdots$ (see Fig. 1). It is then interesting to study their order statistics, where one is concerned with the distribution of $x_k(t)$, which denotes the position of the kth rightmost particle. An equally interesting quantity is the spacing between consecutive particles, $g_k(t) = x_k(t) - x_{k+1}(t)$ as well as the density of the particles near the tip of the branching process [9–11]. The questions related to the extremes in this one-dimensional BBM have been studied extensively over the last few decades [4,7–10,12–14]. More recently, extreme statistics in this system have found new applications in the context of estimating the perimeter and area of the convex hull of two-dimensional epidemic spreads [5].

Indeed BBM is a useful toy model to study the broader question of extreme value statistics (EVS) of correlated random variables, a field that has been growing in prominence in recent years. Several important properties sensitive to rare events can be characterized by EVS in a wide variety of disordered systems [15–17]. Although probability distributions functions (PDFs) of the extreme values of uncorrelated variables are well understood [18], the computation of extreme and near-extreme value distributions for strongly correlated variables constitute important open problems in this field [19,20]. Random walks and Brownian motion have recently proved to be useful laboratories where several exact results concerning EVS of correlated variables can be obtained [11,20,21]. In this context BBM represents a useful model where the relevant random variables (the particle positions at time t) are strongly correlated, and yet exact results concerning the extremes can be obtained. In a recent Letter [11] we briefly discussed some of these results for the critical $b = d$ case. The purpose of the present paper is twofold: (i) to provide a detailed derivation of these exact results for the critical case and (ii) to extend these results to off-critical cases $b \neq d$.

In the supercritical regime ($b > d$), the statistics of the position of the rightmost particle $x_1(t)$ has been studied

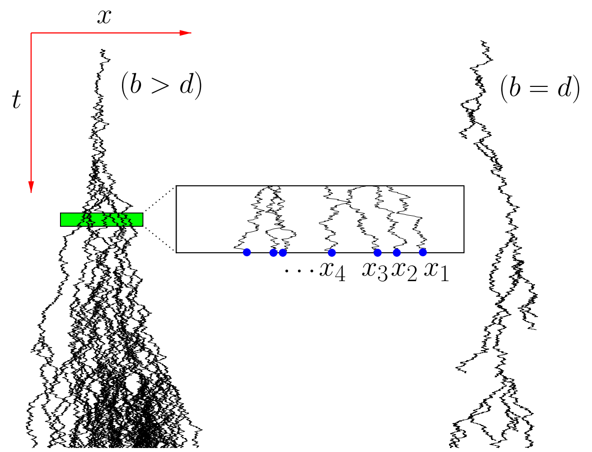

Fig. 1. A realization of the dynamics of branching Brownian motion with death (left) in the supercritical regime ($b > d$) and (right) in the critical regime ($b = d$). The particles are numbered sequentially from right to left as shown in the inset.

for a long time [7,8]. In particular, for the case $d = 0$, the cumulative distribution of $x_1(t)$ is known to be governed by the Fisher–Kolmogorov–Petrovskii–Piscounov (FKPP) equation [1,22]. This equation exhibits a traveling front solution: the average position of the rightmost particle increases linearly with time $\langle x_1(t)\rangle \sim vt$ with a constant velocity v while the width of the front remains of $\mathcal{O}(1)$ at late times. Very recently, Brunet and Derrida studied (still for $d = 0$) the order statistics, i.e., the statistics of the positions of the second, third, etc $x_2(t), x_3(t) \ldots$. They found that, while $x_k(t) \sim vt$ at late times, with the same speed v for all k, the distributions of the gaps $g_k(t)$ become independent of t for large t, while retaining a non-trivial k-dependence [9,10]. They also computed the PDF of the first gap $g_1(t)$ numerically to very high precision and also provided an argument for the observed exponentially decaying tail. Several natural questions remain outstanding. For instance, can one calculate the gap distributions for arbitrary k for $d = 0$ as well as for arbitrary b and d?

As mentioned earlier, in a recent Letter, we were able to compute the order and the gap statistics of BBM at the critical point $b = d$ at a fixed time t, by conditioning the process to have a given number of particles at time t [11]. As we will demonstrate in this paper, this method of conditioning allows us to circumvent the technical difficulties arising from the inherent non-linearities of the problem and provides exact results for arbitrary b and d. Let us briefly summarize our main results. Upon conditioning the system to have exactly n particles at time t, we derive an exact backward Fokker–Planck (BFP) equation for the joint distributions of the ordered positions of the n particles at time t. These equations can, in principle, be solved recursively for all n and the asymptotic results at late times for any fixed n can be extracted explicitly. We find that at large times, and for all b and d, the PDFs of the positions x_k's behave diffusively, $P(x_k, t \to \infty | n) \to \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x_k^2}{4Dt}\right)$, with $k = 1, 2, \ldots$. Note that for $b > d$, this diffusive behavior is in contrast with the case without conditioning on the particle number where it is ballistic. However, as in the case without conditioning, the PDFs of the gaps $g_k(t)$ become stationary in the long time limit. Moreover we show that the stationary gap PDF has an exponential tail in the super-critical ($b > d$) and sub-critical ($b < d$) regimes and an algebraic tail with exponent -3 at the critical point ($b = d$). We argue that these asymptotic tails are *universal* in the sense that they are independent of both n (the particle number) and k (the label of the gap). We also discuss the qualitative differences between the conditioned and unconditioned BBM processes.

The paper is organized as follows. In Section 2, we first compute the mean number of particles at time t after which we show in Section 3 how to compute the statistics of the rightmost particle using a BFP approach. In Section 4, we generalize the BFP approach to compute the (conditional) gap statistics between the two rightmost particles, first in the two-particle sector ($n = 2$), and then for an arbitrary number of particles $n \geqslant 2$. In Section 5, we present an asymptotic analysis of the PDF of the first gap for any n, which we then generalize to the kth gap. In Section 6, we present a comparison of our analytical results with Monte Carlo simulations, before we conclude in Section 7.

2. Number of particles in the system

The number of particles $n(t)$ at time t in the one-dimensional BBM is a random variable, whose distribution can be computed exactly for all b and d. Let $P(n,t)$ be the probability that starting with one particle at time $t=0$, there are exactly n particles at time t. One can derive a backward evolution equation for $P(n,t)$ by considering all microscopic moves that happen in the initial small time interval Δt. In this small interval Δt, the particle either dies with probability $d\Delta t$, splits into two particles with probability $b\Delta t$ and with the remaining probability $1-(b+d)\Delta t$ it diffuses. It is easy to see then that

$$P(n,t+\Delta t) = [1-(b+d)\Delta t]P(n,t) + b\Delta t\sum_{m=0}^{n}P(m,t)P(n-m,t) + d\Delta t\,\delta_{n,0}. \quad (2)$$

By taking the limit $\Delta t \to 0$, this reduces to a partial differential equation

$$\frac{\partial P(n,t)}{\partial t} = -(b+d)P(n,t) + b\sum_{m=0}^{n}P(m,t)P(n-m,t) + d\,\delta_{n,0}. \quad (3)$$

This Eq. (3) can be solved by a standard generating function technique. One gets the following explicit solutions:

$$P(0,t) = \frac{d(e^{bt}-e^{dt})}{be^{bt}-de^{dt}},$$

$$P(n\geq 1,t) = (b-d)^2 e^{(b+d)t}\frac{b^{n-1}(e^{bt}-e^{dt})^{n-1}}{(be^{bt}-de^{dt})^{n+1}}. \quad (4)$$

The average number of particles in the system at a particular time t is then

$$\langle n(t)\rangle = \sum_{n=1}^{\infty} nP(n,t) = e^{(b-d)t}. \quad (5)$$

When $b>d$ the number of particles grows exponentially, whereas when $b<d$ the average number of particles decreases to zero exponentially with time. Exactly at the critical point $b=d$, $\langle n(t)\rangle = 1$ for all t.

Note that, at the critical point, $P(n,t)$ is given by

$$P(0,t) = \frac{bt}{1+bt}, \quad P(n\geq 1,t) = \frac{(bt)^{n-1}}{(1+bt)^{n+1}}. \quad (6)$$

Hence, for large t, the probability to have $n>0$ particles decays to zero as a power law $P(n>0,t)\sim 1/t^2$ while the probability of having no particles approaches to unity also as a power law $P(0,t)\sim 1-1/(bt)$. In this critical case, although the system becomes empty of particles almost surely, the average number of particles remains unity at all times. This indicates that rare events dominate the average behavior and that large fluctuations play a rather important role.

We conclude this section by a remark on the behavior of the *typical* number of particles $n_{\rm typ}(t)$ present in the system at time t, when t is large. This typical number can be estimated by analyzing the probability $P(n\geqslant 1,t)$ in Eqs. (4) and (6) for large n and large t. In the supercritical case, $b>d$, one can show that when $n\to\infty, t\to\infty$, keeping the ratio $n/e^{(b-d)t}$ fixed, $P(n,t)$ takes the scaling form (for $b>d$):

$$P(n,t) \sim \frac{[1-P(0,t\to\infty)]}{e^{(b-d)t}} f_{\rm sup}\left(\frac{n}{e^{(b-d)t}}\right), \quad f_{\rm sup}(y) = \frac{b-d}{d}e^{-\frac{b-d}{d}y}, \quad (7)$$

where $1-P(0,t\to\infty) = (b-d)/b$. Hence the scaling form in Eq. (7) shows that $n_{\rm typ}(t)\sim e^{(b-d)t}$ which coincides in this case with the average number of particles $\langle n(t)\rangle$ given in Eq. (5). On the other hand, at the critical point $b=d$ the situation is quite different. Indeed, in this case, we obtain from Eq. (6) that in the limit $n\to\infty, t\to\infty$ and keeping the ratio $n/(bt)$ fixed, one has

$$P(n,t) \sim \frac{[1-P(0,t)]}{bt} f_{\rm crit}\left(\frac{n}{bt}\right), \quad f_{\rm crit}(x) = e^{-x}. \quad (8)$$

Hence this scaling form (8) shows that the typical number of particles grow linearly with t, $n_{\rm typ}(t)\sim bt$, at variance with the mean number which is $\langle n(t)\rangle = 1$ in this case (5). Finally, in the subcritical case, $d>b$, one has $P(n\geqslant 1,t)\sim(1-P(0,t))\frac{d-b}{b}(b/d)^n$, which shows that in this case $n_{\rm typ}(t)\sim\mathcal{O}(1)$ while the average value decays exponentially $\langle n(t)\rangle \sim e^{-(d-b)t}$ (5). Hence the large time behavior of $n_{\rm typ}(t)$ can be summarized as follows

$$n_{\rm typ}(t) \sim \begin{cases} e^{(b-d)t}, & b>d \\ bt, & b=d \\ \text{const.}, & b<d, \end{cases} \quad (9)$$

which will be useful later.

3. Statistics of the rightmost particle

We begin by analyzing the behavior of the rightmost particle in the system at time t. For this purpose it is convenient to introduce $C(n,x,t)$, denoting the joint probability that there are n particles in the system at time t, and that all the particles are to the left of x. The probability $C(0,x,t)$ does not have any clear meaning, but for convenience we choose $C(0,x,t) = P(0,t)$. The conditional probability that all the particles lie to the left of x, conditioned on the fact that there are exactly n particles at time t is given by $Q(x,t|n) = C(n,x,t)/P(n,t)$, where $P(n,t)$ is given in Eq. (4). The PDF of the position of the rightmost particle is then given by $P(x,t|n) = \frac{\partial}{\partial x}Q(x,t|n)$. By definition $Q(x,t|n)$ satisfies the boundary conditions $Q(x\to\infty,t|n) = 1$ and $Q(x\to-\infty,t|n) = 0$. Initially, since the process starts with a single particle at the origin, it is evident that $P(n,0) = \delta_{n,1}$ and $C(n,x,0) = \delta_{n,1}\theta(x)$, where $\theta(x)$ is the Heaviside theta function. Consequently, the initial condition for the conditional probability is given by $Q(x,0|n) = \theta(x)$ for $n>1$. For $n=0$, we recall that $Q(x,0|0) = 1$ by our convention.

3.1. Backward Fokker–Planck equation for $C(n,x,t)$

In this subsection, we start by deriving a BFP equation for the joint probability $C(n,x,t)$. To see how $C(n,x,t)$ evolves into $C(n,x,t+\Delta t)$ in a small time interval Δt, we split the time interval $[0,t+\Delta t]$ into two subintervals:

$[0, \Delta t]$ and $[\Delta t, t + \Delta t]$. The system first evolves from its initial condition to a new configuration at time Δt which then acts as a new initial condition for the subsequent evolution of duration t over the second subinterval $[\Delta t, t + \Delta t]$. We next enumerate the probabilities of all the events that take place in the first subinterval $[0, \Delta t]$ (see Fig. 2). In this subinterval $[0, \Delta t]$, the particle initially at $x = 0$:

(**A**) dies with probability $d\Delta t$, leading to $n = 0$ particles at all subsequent times. The contribution to the probability $C(n, x, t + \Delta t)$ from this term is then $d\Delta t\, \delta_{n,0}$.

(**B**) splits with probability $b\Delta t$, resulting in two particles at $x = 0$. These two particles give rise to two independent sub-trees. Let r and $n - r$ denote the number of particles in the left and the right sub-trees respectively, where $0 \leqslant r \leqslant n$. Using the independence of the sub-trees, the net contribution from this event to $C(n, x, t + \Delta t)$ is $b\Delta t \sum_{r=0}^{n} C(r, x, t) C(n - r, x, t)$.

(**C**) diffuses with probability $1 - (b + d)\Delta t$, moving a distance $\Delta x = \eta(0)\Delta t$ in the first time step. This effectively shifts the entire process by a distance Δx. The contribution from this term is then $(1 - (b + d)\Delta t)\langle C(n, x - \eta(0)\Delta t, t)\rangle_{\eta(0)}$. Here, and in the following, $\langle \ldots \rangle_{\eta(0)}$ denotes an average over all possible values of the diffusive jump at the first time step.

Adding the contributions from terms (**A**), (**B**) and (**C**), we arrive at

$$C(n, x, t + \Delta t) = (1 - (b + d)\Delta t)\langle C(n, x - \eta(0)\Delta t, t)\rangle_{\eta(0)} + b\Delta t \sum_{r=0}^{n} C(r, x, t) C(n - r, x, t) + d\Delta t\, \delta_{n,0}. \tag{10}$$

Next, using the properties of the Brownian noise in Eq. (1) we can Taylor expand Eq. (10) up to second order in Δt. Taking the limit $\Delta t \to 0$ we arrive at the backward evolution equation for the cumulative probability

$$\frac{\partial C(n, x, t)}{\partial t} = D \frac{\partial^2 C(n, x, t)}{\partial x^2} - (b + d) C(n, x, t) + b \sum_{r=0}^{n} C(r, x, t) C(n - r, x, t) + d\, \delta_{n,0}. \tag{11}$$

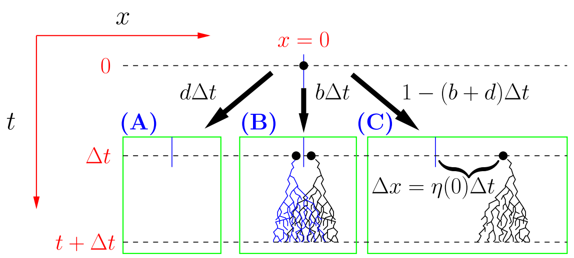

Fig. 2. The backward Fokker–Planck approach: In the first time interval $[0, \Delta t]$, the particle can (**A**) die (**B**) split into two independent particles or (**C**) diffuse by a distance $\Delta x = \eta(0)\Delta t$, with probabilities $d\Delta t, b\Delta t$ and $1 - (b + d)\Delta t$ respectively. We then look at the contribution from each of these events to the probabilities at time $t + \Delta t$.

Using $C(0, x, t) = P(0, t)$ with $P(0, t)$ given in Eq. (4), Eq. (11) reduces to

$$\frac{\partial C(n, x, t)}{\partial t} = D \frac{\partial^2 C(n, x, t)}{\partial x^2} - (b + d) C(n, x, t) + 2bP(0, t) C(n, x, t) + b \sum_{r=1}^{n-1} C(r, x, t) C(n - r, x, t) + d\, \delta_{n,0}. \tag{12}$$

If one sums over the particle number n one gets the cumulative probability distribution of the rightmost particle for the *unconditioned* BBM: $F(x, t) = \sum_{n=0}^{\infty} C(n, x, t)$. Summing Eq. (12) over n one recovers

$$\frac{\partial F(x, t)}{\partial t} = D \frac{\partial^2 F(x, t)}{\partial x^2} - (b + d) F(x, t) + bF^2(x, t) + d, \tag{13}$$

together with the boundary conditions $F(x \to +\infty, t) = 1$ and $F(x \to -\infty, t) = 0$, for all time t. For $d > b$ (super-critical phase) the above equation belongs to the FKPP type of non-linear equations [1,22] which allow for a traveling front solution at late times $F(x, t) \to F(x - vt)$ with a well defined front velocity v [7,8]. In contrast, for $b = d$ (in the critical case), one can show that the solution of (13) is diffusive at late times (the non-linearities give rise to only sub-leading corrections). Unfortunately, for finite t, this non-linear Eq. (13) is not exactly solvable. In contrast, by restricting ourselves to a fixed particle number n sector (without summing over n) we obtain a set of linear equations in $C(n, x, t)$ (12). For any given n the terms in the right hand side of Eq. (12) involve the solution $C(m, x, t)$ with $m < n$. Hence, one can solve these linear equations recursively starting from $n = 1$, for all t and for all b and d. That is the trade-off in order to avoid the non-linearities.

3.2. Late time behavior of the conditional probability $Q(x, t|n)$

Using Eq. (12) for $C(n, x, t)$ and Eq. (4) for $P(n, t)$ one can then write the evolution equation for the conditional probability $Q(x, t|n) = C(n, x, t)/P(n, t)$ explicitly. To proceed, it is convenient to first define

$$f(t) = 2bP(0, t) - (b + d) = (d - b) \frac{be^{bt} + de^{dt}}{be^{bt} - de^{dt}}. \tag{14}$$

We can then remove the linear term in Eq. (12) by making the transformation

$$C(n, x, t) = e^{\int f(t')dt'} C^{\circ}(n, x, t) = \frac{e^{(b+d)t}}{(be^{bt} - de^{dt})^2} C^{\circ}(n, x, t). \tag{15}$$

Inserting this expression into Eq. (12), we arrive at

$$\frac{\partial C^{\circ}(n, x, t)}{\partial t} = D \frac{\partial^2 C^{\circ}(n, x, t)}{\partial x^2} + \frac{be^{(b+d)t}}{(be^{bt} - de^{dt})^2} \sum_{r=1}^{n-1} C^{\circ}(r, x, t) C^{\circ}(n - r, x, t). \tag{16}$$

Next, using Eq. (15) and the expression for $P(n, t)$ in Eq. (4) one gets

$$Q(x,t|n) = \frac{C(n,x,t)}{P(n,t)} = \frac{1}{(b-d)^2}\left(\frac{be^{bt}-de^{dt}}{b(e^{bt}-e^{dt})}\right)^{n-1} C^{\circ}(n,x,t). \tag{17}$$

The evolution equation for $Q(x,t|n)$ can then be finally written as

$$\frac{\partial Q(x,t|n)}{\partial t} = D\frac{\partial^2 Q(x,t|n)}{\partial x^2} + \frac{(b-d)^2 e^{(b+d)t}}{(e^{bt}-e^{dt})(be^{bt}-de^{dt})} \times \sum_{r=1}^{n-1}[Q(x,t|r)Q(x,t|n-r) - Q(x,t|n)]. \tag{18}$$

As we noted before, this is a linear diffusion equation for any n that involves the solutions of $r < n$ as source terms. This set of equations can then be solved recursively to obtain the exact solutions for any n. For example, inserting $n = 1$ in the above equation, we find that $Q(x,t|1)$ obeys the simple diffusion equation without any source for all t, and has the following exact solution

$$Q(x,t|1) = \frac{1}{2}\mathrm{erfc}\left(\frac{-x}{\sqrt{4Dt}}\right), \tag{19}$$

where $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du$ is the complementary error function. The corresponding PDF of the position of the particle conditioned on the event $n = 1$ at time t is then

$$P(x,t|1) = \frac{\partial}{\partial x}Q(x,t|1) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right). \tag{20}$$

We thus find that, for $n = 1$, the solution is purely diffusive at *all times*. In order to analyse the large time behavior for general n in Eq. (18), we note that the cumulative probability is bounded for all x and t ($0 < Q(x,t|n) < 1$). Therefore, at large t, the source term in Eq. (18) tends to zero as $\sim e^{-|b-d|t}$ (for $b \neq d$), and $\sim 1/(bt^2)$ (for $b = d$). Thus, at large times $Q(x,t|n)$ obeys the simple diffusion equation for all $n \geqslant 1$ and the solution behaves for large t as

$$Q(x,t|n) \sim \frac{1}{2}\mathrm{erfc}\left(\frac{-x}{\sqrt{4Dt}}\right), \tag{21}$$

independently of n. From this one can deduce that the PDF of the rightmost particle is diffusive at large times. By symmetry, the leftmost particle also behaves diffusively, and indeed one can show that all the particles confined between these two extreme values behave diffusively at large times with $P(x_k,t|n) \sim \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x_k^2}{4Dt}\right)$ for all $1 \leqslant k \leqslant n$.

Let us comment on this result which may seem counter-intuitive at first sight, especially in the super-critical phase. As described before, in the super-critical phase ($b > d$), the position of the maximum of BBM has a traveling front structure, with the position of the rightmost particle increasing linearly with time $x_1(t) \sim vt$. The effect of conditioning this process on the number of particles n is thus rather drastic in the super-critical phase: it slows down the motion of the rightmost particle from ballistic to diffusive. This can be understood very simply. Without conditioning, the number of particles typically grows exponentially as $n_{\text{typ}}(t) \sim e^{(b-d)t}$ [see Eq. (9)] in the supercritical regime. Upon conditioning to fix n, one picks up contributions only from atypical diffusive trajectories, out of all the possible trajectories up to time t. This can be seen more quantitatively on the equation for $Q(x,t|n)$ in Eq. (18). Indeed in this case the source term in (18) is of the order of $e^{-(b-d)t} \times n$. When analyzing the conditioned process we have neglected this term which is exponentially decaying with t, for fixed n. However, to describe the unconditioned process, one should evaluate $Q(x,t|n_{\text{typ}}(t) = e^{(b-d)t})$ (9) and in this case the source terms $\propto e^{-(b-d)t} \times n_{\text{typ}}(t)$ become of order $\mathcal{O}(1)$ and they can not be neglected. As a consequence the conditioned and the unconditioned process behaves differently. On the other hand, in the critical case $b = d$, this source terms behave like $n_{\text{typ}}(t)/t^2$ and given that in this case $n_{\text{typ}}(t) \sim bt$ (9) it can still be neglected compared to the Laplacian term in the first line of Eq. (18). Hence that is the reason why conditioning on a fixed number of particles allows us to correctly describe the typical late time behavior of the system [11] [note that the same conclusion also holds in the subcritical case where $n_{\text{typ}}(t) \sim \mathcal{O}(1)$, see Eq. (9)].

We note that, although the individual behavior of the particles is diffusive, they are strongly correlated. In order to understand these correlations, we study the gaps between the successive particles. For uncorrelated diffusive particles these gaps would also display a diffusive behavior. However in BBM, quite remarkably as we show in the next section, the PDFs of these gaps become stationary at large times.

4. Gap statistics

We next consider the gap statistics for the conditioned BBM process with $n \geqslant 2$ particles. Let $g_1(t) = x_1(t) - x_2(t)$ denote the gap between the two rightmost particles. To compute the PDF of $g_1(t)$, we study the joint PDF $P(n,x_1,x_2,t)$ that there are exactly n particles ($n \geqslant 2$) at time t, with the first particle at position x_1 and the second at position $x_2 < x_1$. We start with the simplest case $n = 2$ which turns out to be already nontrivial.

4.1. Two-particle sector ($n = 2$)

4.1.1. Backward Fokker–Planck equation for $P(2,x_1,x_2,t)$

We first derive the equation governing the temporal evolution of $P(2,x_1,x_2,t)$ using a similar BFP approach already discussed in Section 3.1. As before, we split the interval $[0,t+\Delta t]$ into two subintervals $[0,\Delta t]$ and $[\Delta t,t+\Delta t]$ (see Fig. 2). In the first subinterval $[0,\Delta t]$, the particle at $x = 0$:

(**A**) dies with probability $d\Delta t$, leading to no particle at subsequent times and thus not contributing to the probability $P(2,x_1,x_2,t)$.

(**B**) splits into two particles with probability $b\Delta t$. Here there are two distinct cases to consider (see Fig. 3):

(i) one branch gives rise to a single particle at the final time at position x_1 and the other gives rise to a single particle at position x_2. The contribution from this term is then $2b\Delta t P(1,x_1,t)P(1,x_2,t)$ where $P(1,x,t)$ is the PDF of having exactly one particle at time t at position x. The combinatorial factor 2 comes from interchanging the two branches. Note that $P(1,x,t)=\partial_x C(1,x,t)$ where $C(1,x,t)=P(1,t)Q(x,t|1)$ with $P(1,t)$ given in Eq. (4) and $Q(x,t|1)$ given in Eq. (19) respectively. This gives explicitly

$$P(1,x,t)=(b-d)^2\frac{e^{(b+d)t}}{\left(be^{bt}-de^{dt}\right)^2}\frac{1}{\sqrt{4\pi Dt}}\times\exp\left(-\frac{x^2}{4Dt}\right). \tag{22}$$

(ii) one branch gives rise to two particles at positions x_1 and x_2 at the final time and the other gives rise to no particle. The contribution from this term is then $2b\Delta t P(0,t)P(2;x_1,x_2,t)$.

(**C**) diffuses by a distance $\Delta x=\eta(0)\Delta t$ with probability $1-(b+d)\Delta t$. Thus for the second subinterval $[\Delta t, t+\Delta t]$, the process starts from the initial position $\Delta x=\eta(0)\Delta t$. Hence the contribution from this term is $(1-(b+d)\Delta t)P(2,x_1-\eta(0)\Delta t,x_2-\eta(0)\Delta t,t)\rangle_{\eta(0)}$.

Adding the contributions from the terms (**A**), (**B**) and (**C**) we arrive at

$$\begin{aligned}P(2,x_1,x_2,t+\Delta t)=&\,(1-(b+d)\Delta t)\langle P(2,x_1\\&-\eta(0)\Delta t,x_2-\eta(0)\Delta t,t)\rangle_{\eta(0)}\\&+2b\Delta tP(0,t)P(2,x_1,x_2,t)\\&+2b\Delta tP(1,x_1,t)P(1,x_2,t).\end{aligned}\tag{23}$$

Expanding the above equation up to second order in Δt, using the properties of the noise in Eq. (1) and taking the limit $\Delta t\to 0$, we arrive at the following evolution equation for the PDF

$$\begin{aligned}\frac{\partial}{\partial t}P(2,x_1,x_2,t)=&\,D\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)^2P(2,x_1,x_2,t)\\&+f(t)P(2,x_1,x_2,t)\\&+2bP(1,x_1,t)P(1,x_2,t),\end{aligned}\tag{24}$$

where $f(t)$ is given in Eq. (14).

4.1.2. Exact solution

Remarkably Eq. (24) can be solved exactly for all t, as we now show. First, it is convenient to get rid of the second term on the right hand side of Eq. (24) by the customary transformation

$$P(2,x_1,x_2,t)=\frac{e^{(b+d)t}}{\left(be^{bt}-de^{dt}\right)^2}P^{\circ}(2,x_1,x_2,t). \tag{25}$$

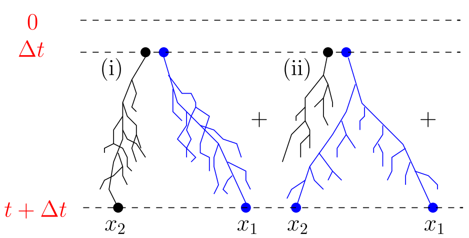

Fig. 3. The contribution from the branching term in the BFP equation for the two-particle sector. The particles at x_1 and x_2 arise from (i) two different offspring (ii) from the same offspring, generated at the first time step.

$P^{\circ}(2,x_1,x_2,t)$ then satisfies

$$\begin{aligned}\frac{\partial}{\partial t}P^{\circ}(2,x_1,x_2,t)=&\,D\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)^2P^{\circ}(2,x_1,x_2,t)\\&+2b\frac{\left(be^{bt}-de^{dt}\right)^2}{e^{(b+d)t}}P(1,x_1,t)P(1,x_2,t).\end{aligned}\tag{26}$$

Next we make the natural change of variables $s=(x_1+x_2)/2$ and $g_1=x_1-x_2>0$ where s denotes the center of mass and g_1 the gap between the two particles. The Jacobian of this transformation is 1. The function $P^{\circ}(2,x_1,x_2,t)$ can be expressed as a function of the new coordinates s and g_1. In order not to proliferate the number of different functions, we denote this function again by $P^{\circ}(2,s,g_1,t)$ and apologise for this slight abuse of notation.

Using the explicit expression for $P(1,x,t)$ from Eq. (22) into Eq. (26), we have

$$\begin{aligned}\frac{\partial}{\partial t}P^{\circ}(2,s,g_1,t)=&\,D\left(\frac{\partial}{\partial s}\right)^2P^{\circ}(2,s,g_1,t)+2b\\&\times\frac{e^{(b+d)t}}{\left(be^{bt}-de^{dt}\right)^2}(b-d)^4\frac{1}{4\pi Dt}\\&\times\exp\left(-\frac{2s^2+\frac{1}{2}g_1^2}{4Dt}\right).\end{aligned}\tag{27}$$

This is a diffusion equation with a time-dependent source term. We recall here that the general diffusion equation with a time-dependent source term

$$\frac{\partial}{\partial t}G(x,t)=D\frac{\partial^2}{\partial x^2}G(x,t)+\sigma(x,t), \tag{28}$$

with a given initial condition $G(x,0)$, can be solved as

$$\begin{aligned}G(x,t)=&\int_{-\infty}^{\infty}\frac{dx'}{\sqrt{4\pi Dt}}\exp\left(-\frac{(x-x')^2}{4Dt}\right)G(x',0)\\&+\int_0^t\frac{dt'}{\sqrt{4\pi D(t-t')}}\int_{-\infty}^{\infty}dx'\exp\left(-\frac{(x-x')^2}{4D(t-t')}\right)\sigma(x',t').\end{aligned}\tag{29}$$

Using Eq. (29) and the initial condition $P^{\circ}(2,s,g_1,t)=0$, we arrive at the following exact solution

$$\begin{aligned}P^{\circ}(2,s,g_1,t)=&\int_0^t dt'\int_{-\infty}^{\infty}ds'\frac{1}{\sqrt{4\pi D(t-t')}}\\&\times\exp\left(-\frac{(s'-s)^2}{4D(t-t')}\right)\times 2b\\&\times\frac{e^{(b+d)t'}}{\left(be^{bt'}-de^{dt'}\right)^2}(b-d)^4\frac{1}{4\pi Dt'}\\&\times\exp\left(-\frac{2s'^2+\frac{1}{2}g_1^2}{4Dt'}\right).\end{aligned}\tag{30}$$

The conditional PDF of the center of mass s and the gap g_1, given that there are exactly two particles in the system at time t, is then given by

$$P(s,g_1,t|2)=\frac{P(2,s,g_1,t)}{P(2,t)}. \tag{31}$$

Using Eq. (25) and the expression for $P(2,t)$ from Eq. (4) we get

$$P(s,g_1,t|2) = \left(\frac{be^{bt} - de^{dt}}{b(b-d)^2(e^{bt} - e^{dt})}\right)P^{\circ}(2,s,g_1,t). \tag{32}$$

Performing the integration with respect to s' in Eq. (30) and using Eq. (32) we arrive at

$$P(s,g_1,t|2) = \frac{(b-d)^2}{2\pi D}\left(\frac{be^{bt} - de^{dt}}{e^{bt} - e^{dt}}\right)\int_0^t dt' \times \frac{e^{(b+d)t'}}{\left(be^{bt'} - de^{dt'}\right)^2}\frac{e^{-\frac{g_1^2}{8Dt'} - \frac{s^2}{2D(2t-t')}}}{\sqrt{t'(2t-t')}}. \tag{33}$$

We note that in the limit $d \to b$ this reduces to the expression derived in [11], for the gap statistics at the critical point $b = d$, since

$$(b-d)^2 \frac{e^{(b+d)t}}{\left(be^{bt} - de^{dt}\right)^2} \to \frac{1}{(1+bt)^2} \quad \text{as } d \to b. \tag{34}$$

Given the exact solution of the conditional joint PDF $P(s,g_1,t|2)$ in Eq. (33) one can derive the marginal distributions of s and g_1 respectively. We start with the center of mass s. By integrating over g_1 in Eq. (33), we have

$$P(s,t|2) = (b-d)^2\left(\frac{be^{bt} - de^{dt}}{e^{bt} - e^{dt}}\right)\int_0^t dt' \times \frac{e^{(b+d)t'}}{\left(be^{bt'} - de^{dt'}\right)^2}\frac{\exp\left(-\frac{s^2}{2D(2t-t')}\right)}{\sqrt{2\pi D(2t-t')}}. \tag{35}$$

The integral in (35) is dominated by the region $t' \to 0$, and therefore the marginal PDF of the center of mass behaves diffusively $\sim \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{s^2}{4Dt}\right)$ for large t. This is consistent with the diffusive behavior of the particles seen in the previous section. Integrating over the center of mass variable s in Eq. (33), we arrive at the marginal PDF of the gap

$$P(g_1,t|2) = (b-d)^2\left(\frac{be^{bt} - de^{dt}}{e^{bt} - e^{dt}}\right)\int_0^t dt' \times \frac{e^{(b+d)t'}}{\left(be^{bt'} - de^{dt'}\right)^2}\frac{\exp\left(-\frac{g_1^2}{8Dt'}\right)}{\sqrt{2\pi Dt'}}. \tag{36}$$

By taking the limit $d \to b$ in Eqs. (35) and (36) we recover the expressions derived at the critical point $b = d$ for the marginal PDFs of the center of mass s and the gap g_1 respectively, previously obtained in Ref. [11].

For arbitrary values of b and d we find from Eq. (36) that the gap distribution becomes stationary at large times $P(g_1, t \to \infty|2) = p(g_1|2)$, where the stationary gap distribution is given by

$$p(g_1|2) = (b-d)^2 \max(b,d)\int_0^\infty dt' \frac{e^{(b+d)t'}}{\left(be^{bt'} - de^{dt'}\right)^2} \times \frac{\exp\left(-\frac{g_1^2}{8Dt'}\right)}{\sqrt{2\pi Dt'}}. \tag{37}$$

Using a saddle point analysis, we can show that the stationary PDF $p(g_1|2)$ has the following asymptotic behavior for $g_1 \gg 1$

$$p(g_1|2) \sim \begin{cases} \frac{|b-d|^{3/2}}{\sqrt{2D}\max(b,d)} \exp\left(-\sqrt{\frac{|b-d|}{2D}}g_1\right), & \text{for } b \neq d, \\ 8\left(\frac{D}{b}\right)g_1^{-3}, & \text{for } b = d. \end{cases} \tag{38}$$

It is interesting to note that the expression for the PDF of the gap in the supercritical case $b > d$ turns out to be exponential. As mentioned above, this behavior was also obtained for the first gap $g_1 = x_1 - x_2$ in the *unconditioned* BBM [10]. For the case $D = 1, b = 1$ and $d = 0$, the tail was shown to be $\exp(-(1+\sqrt{2})g_1)$ for $g_1 \gg 1$, while in the case of the *conditioned* process we find from (38) that $p(g_1|2)$ also decays exponentially albeit with a different rate, namely $p(g_1|2) \sim \exp(-g_1/\sqrt{2})$ (see the paragraph after Eq. (21) for a discussion of the origin of the differences between the two processes). It is interesting to note that the conditioning of the process on n actually *decreases* the correlations between the extreme points, as observing a large gap between the two rightmost particles is more likely in the conditioned process.

4.2. n–particle sectors with n > 2

When we condition the process to have $n > 2$ particles at time t, we compute the first gap by studying the joint PDF $P(n,x_1,x_2,t)$ that there are exactly n particles in the system at time t, with the first at position x_1 and the second at position $x_2 < x_1$. Here we also use the BFP approach to derive an evolution equation for this joint PDF. The main difference arises in the branching term (**B**) at the first time step. For this branching term, and for $n > 2$, there are three distinct cases to consider (instead of two before):

(i) One branch gives rise to no particle while the other gives rise to n particles. The contribution from this term to the final probability is $2b\Delta t P(0,t) P(n,x_1,x_2,t)$. As noted before in Section 3, the combinatorial factor 2 comes from interchanging the two branches.

(ii) One branch gives rise to 1 particle while the other gives rise to $n-1$ particles. The first two particles from the $(n-1)$-particle branch and the particle from the 1-particle branch are ordered as $x_1 > x_2 > x_3$ at the final time step, with any of them belonging to either branch. The contribution of this term is $2b\Delta t \int_{-\infty}^{x_2} dx_3 \sum_{\tau\in S_3} P(1,x_{\tau_1},t)P(n-1,x_{\tau_2},x_{\tau_3},t)$, where we remind that $P(1,x,t)$ is the PDF of having exactly one particle at time t at position x, given in Eq. (22). Here we denote by $\sum_{\tau\in S_N}$ the sum over the permutations τ of N elements with $\tau_i \equiv \tau(i)$ and we use the convention that $P(r,x_i,x_j,t) = 0$ for $i > j$, for any $r \geqslant 2$.

(iii) Finally one branch gives rise to $r \geqslant 2$ particles while the other gives rise to $n - r \geqslant 2$. The contribution of this term is thus

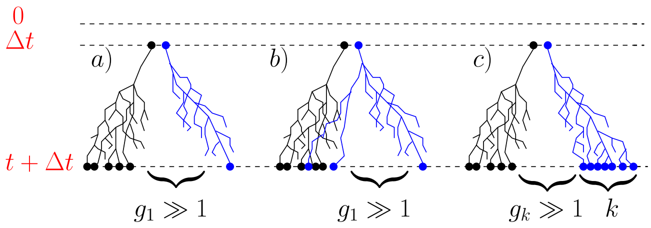

Fig. 4. Dominant terms contributing to the large gap behaviour for (a) the first gap $g_1(t)$ and (c) the k-th gap $g_k(t)$. Figure (b) shows a realization where the large gap is generated by the particles of the same offspring process and is hence suppressed.

$$b\Delta t\sum_{r=2}^{n-2}\int_{-\infty}^{x_2}dx_3\int_{-\infty}^{x_3}dx_4\sum_{\tau\in S_4}P(r;x_{\tau_1},x_{\tau_3},t)P(n-r;x_{\tau_3},x_{\tau_3},t). \tag{39}$$

We can then derive, for any $n>2$, the BFP equation for $P(n,x_1,x_2,t)$, following the same procedure as explained in Section 4.1.1 for the case of $n=2$ particles and obtain:

$$\frac{\partial P(n,x_1,x_2,t)}{\partial t}=D\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)^2P(n,x_1,x_2,t)+f(t)P(n,x_1,x_2,t)+b\,\mathcal{S}(n,x_1,x_2,t), \tag{40}$$

where $f(t)$ is given in Eq. (14) and the source term $\mathcal{S}(n,x_1,x_2,t)$ is obtained by collecting the different contributions computed above:

$$\mathcal{S}(n,x_1,x_2,t)=\int_{-\infty}^{x_2}dx_3\left[2\sum_{\tau\in S_3}P(1,x_{\tau_1},t)P(n-1,x_{\tau_2},x_{\tau_3},t)+\sum_{r=2}^{n-2}\int_{-\infty}^{x_3}dx_4\sum_{\tau\in S_4}P(r,x_{\tau_1},x_{\tau_2},t)P(n-r,x_{\tau_3},x_{\tau_4},t)\right], \tag{41}$$

where $P(1,x,t)$ is given in Eq. (22). We note that while x_1 and x_2 stand for the positions of the first and second particle respectively, x_3 and x_4 are not necessarily the positions of the third and fourth ones.

The BFP equation satisfied by $P(n,x_1,x_2,t)$ (40) and (41) is a linear diffusion equation for any n that involves the solutions for $P(k,x_1,x_2,t)$ for $k<n$. Hence, as noted above in Section 3, this set of equations can be solved recursively to obtain the exact solutions for any n. We have computed these expressions for $n=3$ and 4, but do not present them here as the expressions are rather cumbersome, being expressible as a series of nested integrals. One can show that for any n, the PDF of the first gap $g_1=x_1-x_2$ becomes stationary at large times, $P(g_1,t\to\infty|n)\to p(g_1|n)$, which we study below in the large g_1 limit.

5. Asymptotic behavior

Although, the exact expression of the gap distribution $P(g_1,t|n)$ is a bit cumbersome for arbitrary large values of n, one can analyze its large t and large g_1 limit, from Eqs. (40) and (41) as follows. The solution of (40) is a linear combination of solutions of individual terms in the source function $\mathcal{S}$ in (41). From this, it can be shown that the PDF of the first gap conditioned on n particles converges to a stationary distribution $P(g_1,t\to\infty|n)=p(g_1|n)$. While the

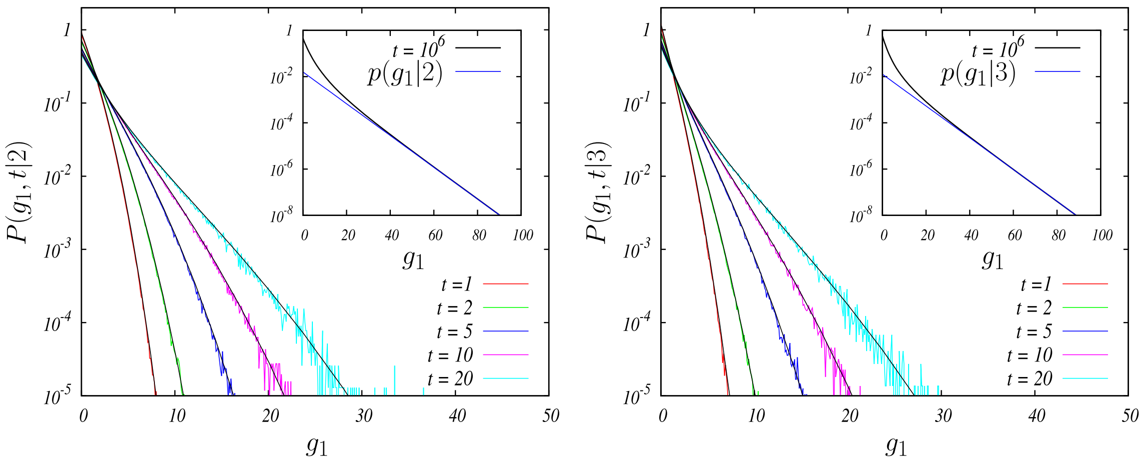

Fig. 5. The marginal PDF of the first gap $g_1=x_1-x_2$ conditioned on **Left** two particles $P(g_1,t|2)$ and **Right** three particles $P(g_1,t|3)$, at different times obtained from Monte Carlo simulations. The black lines correspond to the exact theoretical PDFs (given in Eq. (36) for two particles, the three particle solution was not given here explicitly as it is rather cumbersome). Here $b=0.5, d=0.45$ and $D=1$. These data have been obtained by averaging over 10^7 realizations. In the **Insets** we plot the theoretical PDFs showing the stationary distribution at a late time $t=10^6$, along with the predicted large gap asymptotic behaviour given in Eq. (38).

full PDF $p(g_1|n)$ in general depends on n, its tail is independent of n. This can be seen from the fact that the leading contribution to $\mathcal{S}$ in (41) when the gap $g_1 = x_1 - x_2 \gg 1$ is large arises from the term in the first line of (41) [see Fig. 4(a)]

$$2bP(1,x_1,t)\int_{-\infty}^{x_2} dx_3 P(n-1,x_2,x_3,t) = 2bP(1,x_1,t)P(n-1,x_2,t), \quad (42)$$

where $P(n-1,x_2,t) = \partial_{x_2} C(n-1,x_2,t)$ (we recall that $C(n-1,x_2,t)$ denotes the joint probability that there are $n-1$ particles in the system at time t, and that all the particles are to the left of x_2). Since the rightmost particle conditioned on $n-1$ particles in the system behaves as a free diffusive particle at large times $P(n-1,x_2,t) \sim P(1,x_2,t)$, see Eqs. (17) and (21) like in the $n=1$ - particle case in Eq. (22), we finally obtain that for large t

$$2bP(1,x_1,t)\int_{-\infty}^{x_2} dx_3 P(n-1,x_2,x_3,t) \sim 2bP(1,x_1,t)P(1,x_2,t), \quad (43)$$

which is precisely the source term for the two-particle case analyzed in Eq. (24). This is an advantage of the BFP approach: the two branches arising at the first time step are independent of each other at subsequent times. On the other hand, as we have shown for the two-particle case, the particles from the same branch are strongly correlated at large times. Using this fact, one can show that since all the other terms in $\mathcal{S}$ in (41) involve a larger gap between particles generated by the same branch [see Fig. 4(b)], they are suppressed by a factor $\int_{g_1}^{\infty} p(g'|k)dg'$, $k < n$ which is exponentially small in the supercritical regime and falls as a power-law in the critical regime. Therefore, one has that for large $g_1, p(g_1|n) \sim p(g_1|2)$ independently of $n \geqslant 2$, with the asymptotic behaviors given in Eq. (38).

Similarly the kth gap $g_k(t) = x_k(t) - x_{k+1}(t)$, can be analyzed by studying the joint PDF that there are n particles at time t with the kth particle being at position x_k and the $(k+1)$th particle at position x_{k+1}. This PDF once again satisfies a diffusion equation with a source term similar to (41), from which we can show that the PDF of the kth gap reaches a stationary distribution $P(g_k, t \to \infty|n) = p(g_k|n)$. In the large gap limit, the dominant term in the source function is the one where the first k particles belong to one of the branches generated at the first time step, and the subsequent $n-k$ particles belong to the other [see Fig. 4 c)]. This term tends to $2bP(1,x_k,t)P(1,x_{k+1},t)$ at large t, as it involves the leftmost particle of the first branch being at x_k and the rightmost particle of the other branch being at x_{k+1}. As noticed before for g_1, all other terms involve a large gap between particles generated by the same branch and yield subleading contributions when $g_k \to \infty$. This implies that *the tail of the PDFs of the gaps are universal and are independent of* n *and* k: the large g_k behavior of $p(g_k|n)$ is thus given by Eq. (38) with g_1 replaced by g_k, independently of n. We conclude this section by mentioning that the picture that emerges form our calculation (see Fig. 4) is qualitatively similar to the one of a "clustered" Poisson process discussed recently, for the supercritical case $d = 0$, in the mathematics literature [12–14].

6. Monte Carlo simulations

Finally, we have performed Monte Carlo simulations of the one-dimensional BBM for different values of the parameters b and d. In Fig. 5 we plot the marginal PDF of the gap *conditioned* on a fixed number n of particles (here $n = 2$ and $n = 3$). We find a very good agreement between our theoretical predictions of the gap PDFs and the distributions extracted from the simulations.

7. Conclusion

To conclude, we have obtained exact analytical results for the gap statistics of the extreme particles of BBM conditioned on the number of particles in the system for the general case when $b \neq d$. We derived backward Fokker–Planck equations governing the distributions of the positions of these extreme particles. The conditioning of the PDFs on the number of particles in the system allowed us to express these evolution equations as a system of linear diffusion equations with source terms, which we could then solve recursively. We have also obtained exact results for the gap statistics, which can be obtained from the joint PDF involving the position of two particles. We emphasize that, in the critical and subcritical cases, the conditioned and unconditioned processes lead to the same asymptotic results for the gap and order statistics. The same is not true in the supercritical case. It will be interesting to extend our analysis to the question of k-point correlation functions, with $k > 2$. In this case one can use a similar procedure to analyze the PDF $P(x_1, x_2, x_3, \ldots, t|n)$ that given there are exactly n particles in the system at time t, they are at positions $x_1, x_2, x_3 \ldots$. The solutions can in principle be obtained in the recursive manner as outlined in our paper.

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