Universal Order and Gap Statistics of Critical Branching Brownian Motion

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(Received 18 March 2014; published 29 May 2014)

We study the order statistics of one-dimensional branching Brownian motion in which particles either diffuse (with diffusion constant D), die (with rate d), or split into two particles (with rate b). At the critical point $b = d$, which we focus on, we show that at large time t the particles are collectively bunched together. We find indeed that there are two length scales in the system: (i) the diffusive length scale $\sim\sqrt{Dt}$, which controls the collective fluctuations of the whole bunch, and (ii) the length scale of the gap between the bunched particles $\sim\sqrt{D/b}$. We compute the probability distribution function $\tilde{P}(g_k, t|n)$ of the kth gap $g_k = x_k - x_{k+1}$ between the kth and $(k+1)$th particles given that the system contains exactly $n > k$ particles at time t. We show that at large t, it converges to a stationary distribution $\tilde{P}(g_k, t \to \infty|n) = p(g_k|n)$ with an algebraic tail $p(g_k|n) \sim 8(D/b)g_k^{-3}$, for $g_k \gg 1$, independent of k and n. We verify our predictions with Monte Carlo simulations.

DOI: 10.1103/PhysRevLett.112.210602 PACS numbers: 05.40.Fb, 02.50.Cw, 05.40.Jc

The statistics of the global maximum of a set of random variables finds applications in several fields, including physics, engineering, finance, and geology [1], and the study of such extreme value statistics (EVS) has been growing in prominence in recent years [2–7]. In many real-world examples where EVS is important, the maximum is not independent of the rest of the set, and there are strong correlations between near-extreme values. Examples can be found in meteorology where extreme temperatures are usually part of a heat or cold wave [8] and in earthquakes and financial crashes where extreme fluctuations are accompanied by foreshocks and aftershocks [9–15]. Near-extreme statistics also play a vital role in the physics of disordered systems where energy levels near the ground state become important at low but finite temperature [4]. In this context, the distribution of the kth maximum x_k of an ordered set $\{x_1 > x_2 > x_3 \ldots\}$ (order statistics [16]) and the gap between successive maxima $g_k = x_k - x_{k+1}$ provides valuable information about the statistics near the extreme value. Such near-extreme distributions have recently been of interest in statistics [17] and physics [18–21]. Although the order and gap statistics of independent identically distributed variables are fully understood [16], very few exact analytical results exist for strongly correlated random variables. In this context, random walks and Brownian motion offer a fertile arena where near-extreme distributions for correlated variables can be computed analytically [19–21].

Another interesting system where order statistics plays an important role is the branching Brownian motion (BBM). In BBM, a single particle starts initially at the origin. Subsequently, in a small time interval dt, the particle splits into two independent offsprings with probability $b\,dt$ and dies with probability $d\,dt$, and with the remaining probability $[1 - (b + d)dt]$, it diffuses with diffusion constant D. A typical realization of this process is shown in Fig. 1. BBM is a prototypical model of evolution but has also been extensively used as a simple model for reaction-diffusion systems, disordered systems, nuclear reactions, cosmic ray showers, and epidemic spreads, among others [22–38]. In one dimension, the positions of the existing particles at time t constitute a set of strongly correlated variables that are naturally ordered according to their positions on the line with $x_1(t) > x_2(t) > x_3(t)\ldots$. The particles are labeled sequentially from right to left as shown in Fig. 1. One-dimensional BBM then provides a natural setting to study the order and the gap statistics for strongly correlated variables. Note that the positions $x_i(t)$ in one dimension do not necessarily correspond to a physical distance, but may represent for instance the degree of

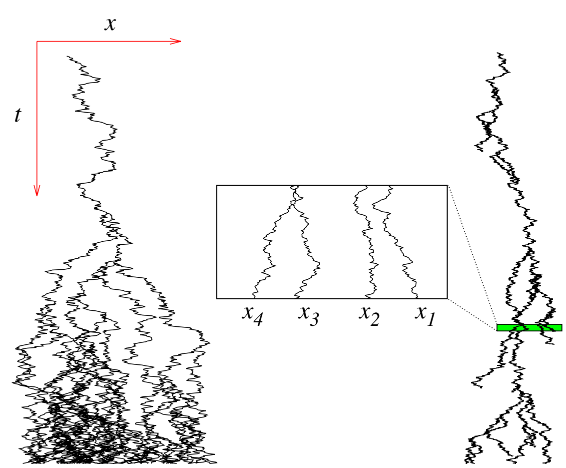

FIG. 1 (color online). A realization of the dynamics of BBM with death (left) in the supercritical regime ($b > d$) and (right) the critical regime ($b = d$). The particles are numbered sequentially from right to left, as shown in the inset.

mutation of a trait [36] or the energy levels in a disordered system [22,23].

The number of particles $n(t)$ present at time t in this process is a random variable with different behavior depending on the relative magnitude of the rates of birth b and death d. When $b < d$ (*subcritical* phase), the process dies eventually, and on an average, there are no particles at large times. In contrast, for $b > d$ (*supercritical* phase), the process is explosive, and the average number of particles grows exponentially with time. In the borderline $b = d$ (*critical*) case, the probability $P(n, t)$ of having n particles at time t, starting with a single particle initially, has a well-known expression [39] (a simple derivation is provided in the Supplemental Material [40]),

$$P(0,t) = \frac{bt}{1+bt}, \qquad P(n \geq 1, t) = \frac{(bt)^{n-1}}{(1+bt)^{n+1}}. \quad (1)$$

The probability that there are no particles tends to 1 as $1 - 1/(bt)$, while the probability that there are $n \geq 1$ particles tends to 0 as $1/(bt)^2$. The average number of particles is independent of time with $\langle n(t)\rangle = \sum_{n=1}^{\infty} n \times P(n, t) = 1$. There are thus strong fluctuations at the critical point, which causes most of the realizations of this process to have no particles at large times.

In the supercritical phase, in particular for $d = 0$, the statistics of the kth rightmost particle $x_k(t)$ have been studied extensively in mathematics and physics literature with direct relevance to polymer [32] and spin-glass physics [33]. For example, the position of the rightmost particle $x_1(t) \sim vt$ typically increases linearly with t, and its cumulative distribution satisfies a nonlinear Fisher-Kolmogorov-Petrovky-Piscounov equation [24,41] with a traveling front solution with velocity v [26,27]. The statistics of this rightmost particle, in the supercritical phase, also appear in numerous other applications in mathematics [42,43] and physics [22,23,37]. More recently, the statistics of the gaps between successive particles have also been studied in the supercritical phase [22,23], and the average gap between the kth and $(k+1)$th particle was shown to tend to a k-dependent constant, independent of time t, at large t. The stationary probability distribution function (PDF) of the first gap was also computed numerically, and an analytical argument was given to explain its exponential tail [22,23]. However, an exact analytical computation of the stationary PDFs of these gaps in the supercritical phase still remains an open problem.

Much less is known about the order statistics at the critical point ($b = d$), which is relevant to several systems, including population dynamics, epidemic spreads, nuclear reactions, etc. [37,44–46]. In this Letter, we show that, in contrast to the supercritical case, the order and the gap statistics can be computed exactly for the critical case $b = d$. In the critical case where $\langle n(t)\rangle = 1$ at all times, to make sense of the gaps between particles, it is necessary to condition the process to have exactly $n(t) = n$ particles at time t, with their ordered positions denoted by $x_1(t) > \cdots x_k(t) \cdots > x_n(t)$. We show that a typical trajectory of the critical process is characterized by two length scales at late times: (i) each particle $\langle |x_k(t)|\rangle \sim \sqrt{4Dt/\pi}$ for all $1 \leq k \leq n$, implying an effective bunching of the particles into a single cluster that diffuses as a whole, and (ii) within this bunch, the gap $g_k(t) = x_k(t) - x_{k+1}(t)$ between successive particles tends to a time-independent random variable of $\sim O(1)$. We compute analytically the PDF of this gap (conditioned on n particles) and show that it becomes stationary at late times $\tilde{P}(g_k = z, t \to \infty | n) \to p(z|n)$ independent of k. Moreover, quite remarkably, $p(z|n)$ has a *universal* algebraic tail, $p(z|n) \sim 8(D/b)/z^3$, independent of k and n.

Statistics of the rightmost particle.—We first analyze the behavior of the rightmost particle at time t. A convenient quantity is the joint probability that there are $n \geq 1$ particles at time t, with all of them lying to the left of x: $Q(n;x,t) = \text{Prob.}[n(t) = n, x_n(t) < x_{n-1}(t) < \ldots < x_1(t) < x]$. It evolves via a backward Fokker-Planck (BFP) equation, which can be derived by splitting the time interval [0, $t + \Delta t$] into [0, Δt] and [Δt, $t + \Delta t$] and considering all events that take place in the first small interval [0, Δt]. In this small interval, the single particle at the origin can (i) with a probability $b\Delta t$ split into two independent particles, which give rise to r and $n - r$ particles at the final time, respectively, (ii) die with the probability $d\Delta t$ and therefore not contribute to the probability at subsequent times, or (iii) diffuse by a small amount Δx with probability $1 - (b + d)\Delta t$, effectively shifting the entire process by Δx. Summing these contributions, taking the $\Delta t \to 0$ limit and setting $b = d$, we get (see the Supplemental Material [40])

$$\frac{\partial Q(n;x,t)}{\partial t} = D\frac{\partial^2 Q(n;x,t)}{\partial x^2} - 2bQ(n;x,t) + 2bP(0,t)Q(n;x,t) + b\sum_{r=1}^{n-1} Q(r;x,t)Q(n-r;x,t), \quad (2)$$

starting from the initial condition $Q(n;x,0) = \delta_{n,1}$ for all $x > 0$ and satisfying the boundary conditions: $Q(n;-\infty,t) = 0$ and $Q(n;\infty,t) = P(n,t)$. Next, we consider the conditional probability $Q(x,t|n) = Q(n;x,t)/P(n,t)$, i.e., the cumulative probability of the rightmost particle given n particles at time t. Using (2) and the explicit expression of $P(n,t)$ in (1), we find that $Q(x,t|n)$ evolves via

$$\frac{\partial Q(x,t|n)}{\partial t} + \frac{n-1}{t(1+bt)}Q(x,t|n) = D\frac{\partial^2 Q(x,t|n)}{\partial x^2} + \frac{1}{t(1+bt)}\sum_{r=1}^{n-1} Q(x,t|r)Q(x,t|n-r). \quad (3)$$

This is a linear equation for $Q(x,t|n)$ for a given n that involves, as source terms, the solutions $Q(x,t|k)$ with $k<n$. Hence, it can be solved recursively for any n, starting with $n=1$. For $n=1$, one obtains an explicit solution (see the Supplemental Material [40]), $Q(x,t|1)=\frac{1}{2}\mathrm{erfc}(-x/\sqrt{4Dt})$, where $\mathrm{erfc}(x)=(2/\sqrt{\pi})\int_x^\infty e^{-u^2}du$ is the complementary error function. Consequently, the PDF of $x_1(t)$ conditioned on there being one particle at time t, $P(x_1,t|1)=\partial_{x_1}Q\times(x_1,t|1)=(1/\sqrt{4\pi Dt})\exp(-x_1^2/4Dt)$, is a simple Gaussian, exhibiting free diffusion. For later purposes, we note that $P(1;x,t)=\partial_x Q(1;x,t)=P(1,t)\partial_x Q(x,t|1)$; i.e., the probability density of having one particle at position x at time t reads

$$P(1;x,t)=\frac{1}{(1+bt)^2}\frac{1}{\sqrt{4\pi Dt}}e^{-x^2/4Dt}. \tag{4}$$

Finally, feeding the one particle solution $Q(x,t|1)$ into (3) for $n=2$, one can also obtain $Q(x,t|2)$ (see the Supplemental Material [40]) and recursively $Q(x,t|n)$ for higher n.

For general $n>1$, one can estimate easily the late-time asymptotic solution. Since $Q(x,t|n)$ is bounded as $0<Q(x,t|n)<1$, Eq. (3) reduces, for large t, to a simple diffusion equation, which does not contain n explicitly, implying $Q(x,t|n)\sim Q(x,t|1)$. Hence, the PDF of the rightmost particle conditioned on there being $n\geq 1$ particles at time t behaves as $P(x_1,t|n)\approx(1/\sqrt{4\pi Dt})\times\exp(-x_1^2/4Dt)$ for large t. By symmetry, the leftmost particle x_n is also governed by the same distribution. This illustrates an important feature of BBM at criticality: the rightmost and leftmost particles behave as free diffusing particles at large t. The rest of the particles are confined between these two extreme values $[x_1(t)>\cdots x_k(t)\cdots>x_n(t)]$ and hence also behave diffusively, $\langle|x_k|\rangle\sim\sqrt{4Dt/\pi}$, independent of k and n for large t, leading to the bunching of the particles. The gap between the particles $g_k(t)=x_k(t)-x_{k+1}(t)$ thus probes the subleading large t behavior of the particle positions $x_k(t)$, which we consider next.

Gap statistics.—We start with the first gap $g_1(t)=x_1(t)-x_2(t)$ between the two rightmost particles conditioned on there being $n\geq 2$ particles at time t. To compute this gap, it is convenient to study the joint PDF $P(n;x_1,x_2,t)$ that there are n particles at time t with the first particle at position x_1 and the second at position $x_2<x_1$. We first analyze the simplest case $n=2$ and argue later that the behavior of g_1 for $n=2$ is actually quite generic and holds for higher n as well. Using a similar BFP approach outlined before, we find the following evolution equation (for detailed derivation, see the Supplemental Material [40]):

$$\frac{\partial P(2;x_1,x_2,t)}{\partial t}=D\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)^2P(2;x_1,x_2,t)-\frac{2b}{1+bt}P(2;x_1,x_2,t)+2bP(1;x_1,t)P(1;x_2,t), \tag{5}$$

where $P(1;x,t)$ is given in (4). This linear equation for $P(2;x_1,x_2,t)$ can be solved explicitly [40]. Consequently, the conditional probability $P(x_1,x_2,t|2)=P(2;x_1,x_2,t)/P(2,t)$ [with $P(2,t)=bt/(1+bt)^3$ given in (1)], denoting the joint PDF of x_1 and x_2 given $n=2$ particles, can also be obtained explicitly. The solution is best expressed in terms of the variables, $s=(x_1+x_2)/2$ (center of mass) and $g_1=x_1-x_2$ (gap): $P(x_1,x_2,t|2)\to\tilde{P}(s,g_1,t|2)$ and reads [40]

$$\tilde{P}(s,g_1,t|2)=\left(\frac{1+bt}{2\pi Dt}\right)\int_0^t\frac{dt'}{(1+bt')^2}\times\frac{e^{-(g_1^2/8Dt')-(s^2/2D(2t-t'))}}{\sqrt{t'(2t-t')}}. \tag{6}$$

It is easy to check that the marginal PDF of the center of mass $\int_0^\infty\tilde{P}(s,g_1,t|2)dg_1$ behaves diffusively $\sim(1/\sqrt{4\pi Dt})\times\exp(-s^2/4Dt)$ for large t. Similarly, one can obtain the marginal PDF of the gap $\tilde{P}(g_1,t|2)=\int_{-\infty}^\infty\tilde{P}(s,g_1,t|2)ds$ at any t

$$\tilde{P}(g_1,t|2)=\left(\frac{1+bt}{bt}\right)\int_0^t\frac{bdt'}{(1+bt')^2}\frac{\exp(-\frac{g_1^2}{8Dt'})}{\sqrt{2\pi Dt'}}. \tag{7}$$

At large times, $\tilde{P}(g_1,t|2)$ converges to a stationary distribution $\tilde{P}(g_1,t\to\infty|2)=p(g_1|2)$ (Fig. 2), which can be computed explicitly. It can be expressed as $p(g_1|2)=(4\sqrt{D/b})^{-1}f[g_1/(4\sqrt{D/b})]$ with

$$f(x)=-4x+\sqrt{2\pi}e^{2x^2}(1+4x^2)\mathrm{erfc}(\sqrt{2}x). \tag{8}$$

This distribution (8) has a very interesting relation to the PDF of the (scaled) kth gap between extreme points visited by a single random walker found in Ref. [19] [the scaling

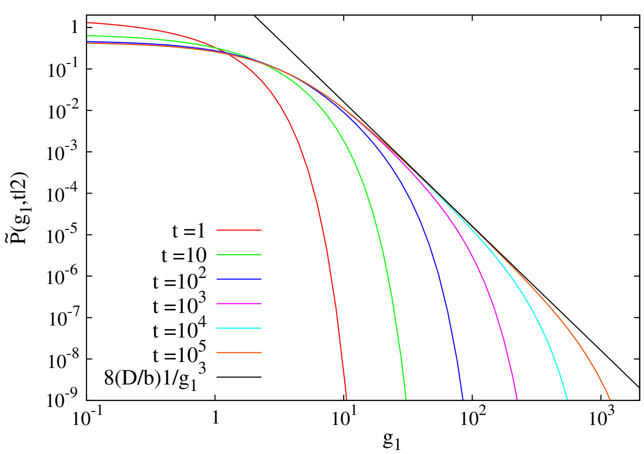

FIG. 2 (color online). Exact gap PDF conditioned on two particles [Eq. (7)] at different times, showing the approach to the stationary behavior at large times. The solid line indicates the expected power law decay for $t\to\infty$. Here $D=1$ and $b=1/2$

function found there, see Eq. (1) of [19], is exactly $-f'(x)/\sqrt{2\pi}$]. It behaves asymptotically as

$$p(g_1|2) \sim \begin{cases} \sqrt{\frac{\pi b}{8D}}, g_1 \to 0, \\ (\frac{8D}{b}) g_1^{-3}, g_1 \to \infty. \end{cases} \quad (9)$$

This function $p(g_1|2)$ describes the typical fluctuations of the gap g_1, which are of order $\sqrt{D/b}$. However, because of the algebraic tail, only the first moment of the gap is dominated by the typical fluctuations, $\langle g_1 \rangle = \sqrt{2\pi D/b}$. The higher moments instead get contributions from the time-dependent far tail of the PDF in (7): $\langle g_1^2 \rangle \sim \ln(t)$ and $\langle g_1^m \rangle \sim t^{(m/2)-1}$ for $m > 2$. In Fig. 2, we plot $\tilde{P}(g_1, t|2)$ at different times, showing the approach to the stationary distribution with a power law tail at large times.

The computation for the first gap g_1 for $n = 2$ outlined above can be generalized to the case when $n > 2$. Once again using the BFP approach, we find that the joint PDF $P(n; x_1, x_2, t)$ obeys

$$\frac{\partial P(n; x_1, x_2, t)}{\partial t} = D\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2 P(n; x_1, x_2, t) - \frac{2b}{1 + bt} P(n; x_1, x_2, t) + b\mathcal{S}(n; x_1, x_2, t). \quad (10)$$

Here $\mathcal{S}(n; x_1, x_2, t)$ is a source term that arises from the branching at the first-time step. It can be computed explicitly in terms of spatial integrals involving $P(k;x_1,x_2,t)$ with $k < n$—the resulting expression being however a bit cumbersome (see the Supplemental Material [40]). Hence, Eq. (10) can in principle be solved recursively to obtain the exact distribution of the first gap $g_1 = x_1 - x_2$ for general n. We have solved these equations exactly up to $n = 4$ [40]. These computations are instructive to analyze Eq. (10) in the large t and large g_1 limit for generic n. Omitting details [40], we find that indeed for general n, the PDF of the first gap tends at late times to a stationary distribution $\tilde{P}(g_1, t \to \infty|n) = p(g_1|n)$. While the full stationary PDF $p(g_1|n)$ depends on n (see Fig. 3), its tail for large g_1 turns out to be universal, $p(g_1|n) \sim (8D/b) g_1^{-3}$, for all n. The asymptotic tail of the PDF of the kth gap conditioned on n particles can be similarly estimated by writing down the evolution equation for $P(n; x_k, x_{k+1}, t)$, the joint PDF of having n particles at time t with the kth particle at x_k and $(k+1)$th particle at x_{k+1}. Analyzing this equation in a similar way (see the Supplemental Material [40]), one concludes (i) $\tilde{P}(g_k, t \to \infty|n) = p(g_k|n)$ (stationary distribution) and (ii) for large g_k, $p(g_k|n) \sim (8D/b) g_k^{-3}$ for all k and n.

Monte Carlo simulations.—We have directly simulated the critical BBM process, and we have computed the PDFs

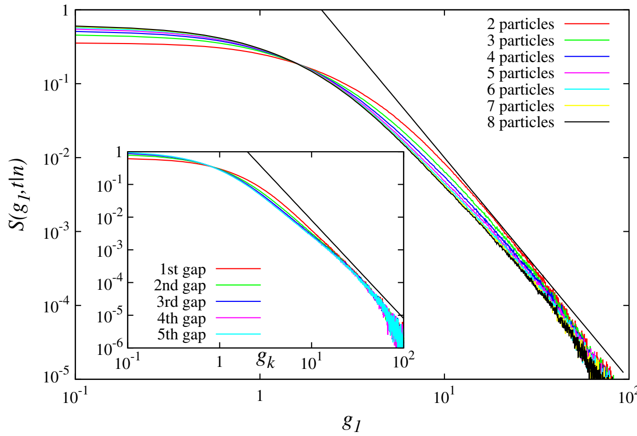

FIG. 3 (color online). Time-integrated PDF for the first gap $g_1 = x_1 - x_2$ conditioned on different numbers of particles computed from Monte Carlo simulations. (Inset) Time-integrated PDF for the kth gap $g_k = x_k - x_{k+1}$ conditioned on ten particles, showing the approach to the same asymptotic value. The lines have a slope of -3. Here $D = 1$, $b = 1/2$, and $t = 10^4$.

of the gap. To obtain better statistics, we compute the time-integrated PDF $S(g_k, t|n) = (1/t) \int_0^t \tilde{P}(g_k, t'|n) dt'$, which has the same stationary behavior as $\tilde{P}(g_k, t|n)$, $S(g_k, t \to \infty|n) = p(g_k|n)$. In Fig. 3, we plot $S(g_1, t|n)$, corresponding to the first gap, for different values of $n = 1, \ldots, 8$ and $t = 10^4$. The different curves show an approach to the same asymptotic, large g_1, behavior (note that the approach to the stationary state gets slower as n increases). In the inset of Fig. 3, we show a plot of $S(g_k, t|n)$ for $n = 10$ and $t = 10^4$ for different values of $k = 1, \ldots, 5$. This also shows a convergence to the same large g_k behavior $\sim (8D/b) g_k^{-3}$. Numerical results for short times (up to $n = 4$), not shown here (see the Supplemental Material [40]), show a perfect agreement with the solution of Eq. (10).

Conclusion.—In this Letter, we obtained exact results for the gap distribution of the critical BBM in one dimension. We circumvented the problem of nonlinearities by conditioning the process to have a fixed particle number. This kind of conditioning is actually quite general and may prove useful in other generic problems involving birth, death, and branching. We showed that the statistics of the near-extreme points display a quite rich behavior characterized by a stationary gap distribution with a universal algebraic tail. It will be interesting to extend this method to exactly compute the gap statistics in the supercritical case. Finally, our method can be easily extended to branching processes where diffusion takes place in higher dimensions with the radial distance undergoing a Bessel process. In this case, one can order the particles by their radial distance from the origin and study their order and gap statistics.

K. R. acknowledges helpful discussions with Shamik Gupta. S. N. M. and G. S. acknowledge support by ANR

Grant No. 2011-BS04-013-01 WALKMAT and in part by the Indo-French Centre for the Promotion of Advanced Research under Project No. 4604-3. G. S. acknowledges support from Labex-PALM (Project Randmat).

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