Supplementary material: Role of subleading terms V_3 and V_4

March 21, 2012

In this supplementary material we explore the convergence properties of the range expansion developed in our Letter by studying the effects of the subleading three-body and four-body interactions V_3 and V_4 in a small 6×6 lattice with open boundary conditions using exact enumeration techniques.

We use a standard backtrack procedure [1] to generate all the allowed configurations of fully packed dimers on a 6 X 6 square lattice with open boundary conditions. We find that the system has $N_C = 6728$ configurations. Using this list of dimer configurations, we generate all possible loop configurations $\mathcal{L}$ by the geometric superposition of any two valid dimer configurations. We compute the exact partition function $\mathcal{Z}_{loop}$ of the loop model by summing over all $\mathcal{L}$ with weight

$$w_{loop}(g, \mathcal{L}) = (g)^{n_d(\mathcal{L})}(2g)^{n_l(\mathcal{L})} , \tag{1}$$

where $n_d(\mathcal{L})$ is the number of doubled edges and $n_l(\mathcal{L})$ the number of non-trivial loops in the loop configuration $\mathcal{L}$.

We are now in a position to compare the exact results for various observables with the predictions obtained from the interacting dimer model. Our interest in doing this is to compare the predictions obtained by truncating the range expansion at the level of two-body interactions with improved estimates obtained by keeping three-body and four-body interactions.

As an illustrative example, we consider

$$Q = \langle [n_h(\mathcal{D}) - n_v(\mathcal{D})]^2 \rangle, \tag{2}$$

where $n_h(\mathcal{D})$ and $n_v(\mathcal{D})$ are the number of horizontal and vertical dimers in the dimer configuration $\mathcal{D}$ respectively. The angular brackets denote the average over the dimer ensemble defined by the weight $\langle \mathbf{\Psi}(g)|\mathcal{D}\rangle$. Using our exact enumeration results for the loop model partition function $\mathcal{Z}_{loop} = \langle \mathbf{\Psi}(g)|\mathbf{\Psi}(g)\rangle$, we compute Q exactly as a function of g for the 6×6 lattice mentioned earlier. We compare Q with interacting dimer model predictions obtained for the same quantity using our exact enumeration of dimer configurations, but keeping only some of the interactions. In this manner, we obtain three different approximations to Q: Q_{V_2}, which is computed using weights that incorporate only two-body interactions, $Q_{(V_2+V_3)}$ computed using two-body and three-body interactions,and $Q_{(V_2+V_3+V_4)}$ computed using two, three and four-body interactions.

For this purpose, we use the expressions for V_2 and V_3 in the main text and work out the corresponding results for V_4. We find that V_4 is zero except in the

following cases

$$V_4\left(\square\square\square\right) = -\log\left(\frac{(1+g^{-1})(1+3g^{-1}+g^{-2})}{(1+2g^{-1})^2}\right) \tag{3}$$

$$V_4\left(\square\square\square\right) = -\log\left(\frac{(1+g^{-1})(1+g^{-1}+2g^{-2}+g^{-3})}{(1+g^{-1}+g^{-2})^2}\right) \tag{4}$$

$$V_4\left(\square\square\square\right) = -\log\left(\frac{(1+2g^{-1}+g^{-2}+g^{-3})}{(1+g^{-1})^2}\right) \tag{5}$$

$$V_4\left(\square\square\square\right) = -\log\left(\frac{(1+2g^{-1}+2g^{-2})}{(1+g^{-1})(1+g^{-1}+g^{-2})}\right) \tag{6}$$

$$V_4\left(\square\square\right) = -\log\left(\frac{(1+g^{-1})(1+2g^{-1}+g^{-2})}{(1+2g^{-1})(1+g^{-1}+g^{-2})}\right) \tag{7}$$

$$V_4\left(\square\square\right) = -\log\left(\frac{(1+g^{-1}+g^{-2}+g^{-3})}{(1+g^{-1}+g^{-2})}\right) \tag{8}$$

and their symmetry related counterparts.

In Fig 1, we display the corresponding results for Q_{V_2}, $Q_{(V_2+V_3)}$, and $Q_{(V_2+V_3+V_4)}$ on the 6×6 sample, and compare them to the exact result Q.

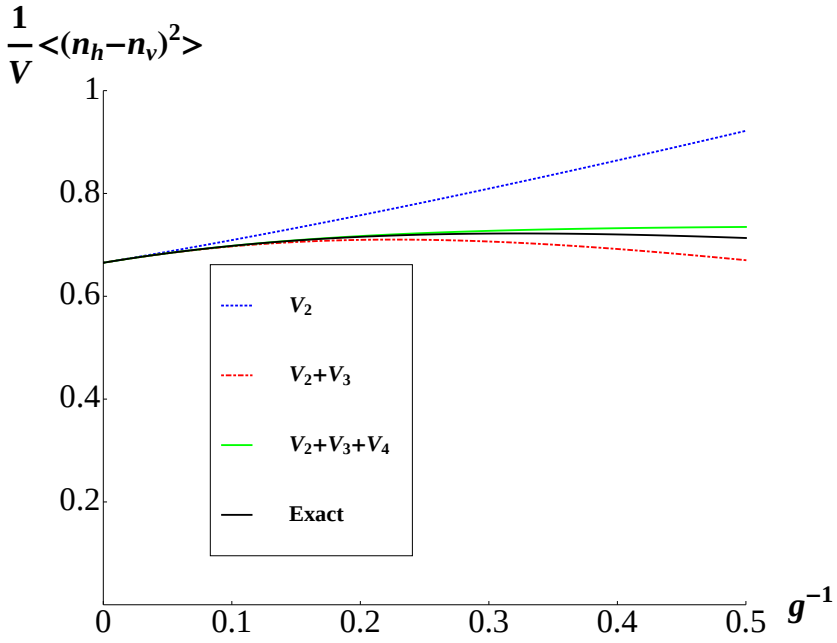

Figure 1: Plot of $Q \equiv \langle (n_h - n_v)^2 \rangle / L^2$ for a $L = 6$ open lattice as a function of g^{-1}, compared with Q_{V_2}, $Q_{(V_2+V_3)}$, and $Q_{(V_2+V_3+V_4)}$.

We find that the result using only V_2 overestimates the answer, while adding the next order term underestimates it. V_4 once again gives a larger estimate of the answer compared to the exact answer. This illustrates the important

oscillatory nature of the range expansion, namely the fact that V_2 and V_3 have opposite effects: The former favours columnar order of dimers, but overestimates it for not too small g^{-1}. V_3 corrects this. This correction is asymptotically exact for $g^{-1} \to 0$, but at finite g^{-1} it leads to a bit of overcorrection. This is rectified by V_4, and so on. For observables which take on a large value in a columnar ordered phase and are suppressed in a disordered phase, this can in principle lead to non-monotonic convergence of the range expansion. Q is an example of such a variable, and the observed non-monotonic convergence to the exact result is an illustration of this general point.

With this in mind, we turn to our estimate for α obtained in our Letter by including only two-body interactions. Since the two-body interactions favour columnar order and the three-body interactions disfavour it, we expect that V_3 will most likely *decrease* the effective stiffness $\kappa(g)$ relative to its value in a system with only two-body interactions present. As a result, we expect that the inclusion of V_3 will result in a corresponding *increase* in our estimate of $\alpha(g) \equiv 1/\kappa(g)$. This will result in an estimate for $\alpha(g = 2)$ that is slightly *greater* than the leading order estimate $\alpha(g = 2) \approx 1.22$, which we have obtained in our Letter by including only two-body interactions.

In other words, we expect that inclusion of V_3 will actually *worsen* the agreement with the numerically determined exact value $\alpha(2) \approx 1.20\ldots$ [2, 3]. Moreover, we expect that including V_4 in addition to V_2 and V_3 will again bring our estimate closer to the exact value, similar to the non-monotonic convergence seen in our finite-size example discussed above.

References

[1] S. Redner, J. Stat. Phys., 29 (1982), p. 309

[2] A. F. Albuquerque and F. Alet, Phys. Rev. B **82**, 180408(R) (2010).

[3] Y. Tang, A. W. Sandvik, and C. L. Henley, Phys. Rev. B **84**, 174427 (2011).